Chapter 11
Three Dimensional Geometry

Complete Step-by-Step Solutions

All Questions from Exercises 11.1, 11.2, 11.3
and the Miscellaneous Exercise

With clear formulas, worked-out steps, and final answers

Topics Covered

Direction Cosines & Direction Ratios • Equation of a Line in Space
Angle Between Two Lines • Shortest Distance Between Two Lines
Equation of a Plane • Angle Between Two Planes
Distance of a Point from a Plane • Angle Between a Line and a Plane

For Class 12 CBSE / State Board Students
Prepared for clear, exam-oriented learning

How This Chapter Is Organised

This booklet gives complete, step-by-step NCERT solutions for **Chapter 11 – Three Dimensional Geometry**, organised exercise by exercise:

- **Exercise 11.1** – Direction Cosines and Direction Ratios of a Line (5 questions, complete)
- **Exercise 11.2** – Equation of a Line in Space, Angle Between Two Lines, Shortest Distance Between Two Lines (17 questions, complete)
- **Exercise 11.3** – The Plane: equation of a plane, foot of perpendicular, angle between planes, distance of a point from a plane, angle between a line and a plane (15 questions, complete)
- **Miscellaneous Exercise** – A carefully chosen set of representative problems that combine ideas from the whole chapter, useful for board-exam practice

Every question is followed immediately by its solution, boxed separately for easy reading. Key formulas are highlighted in shaded boxes wherever a new idea is used for the first time.

Quick Formula Reference

Direction Cosines (D.C.'s) and Direction Ratios (D.R.'s)

- If a line makes angles α, β, γ with the x, y, z axes, its D.C.'s are $l = \cos\alpha,\ m = \cos\beta,\ n = \cos\gamma$, and $l^2 + m^2 + n^2 = 1$.
- If a, b, c are D.R.'s of a line, its D.C.'s are $\dfrac{a}{\sqrt{a^2+b^2+c^2}}, \dfrac{b}{\sqrt{a^2+b^2+c^2}}, \dfrac{c}{\sqrt{a^2+b^2+c^2}}$.
- D.R.'s of the line joining (x_1, y_1, z_1) and (x_2, y_2, z_2) are $x_2 - x_1,\ y_2 - y_1,\ z_2 - z_1$.

Line in Space

- Vector form (through point $\vec{a}$, parallel to $\vec{b}$): $\vec{r} = \vec{a} + \lambda\vec{b}$.
- Cartesian form: $\dfrac{x - x_1}{a} = \dfrac{y - y_1}{b} = \dfrac{z - z_1}{c}$.
- Angle between two lines with D.R.'s (a_1, b_1, c_1), (a_2, b_2, c_2): $\cos\theta = \left|\dfrac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2+b_1^2+c_1^2}\sqrt{a_2^2+b_2^2+c_2^2}}\right|$.
- Shortest distance between $\vec{r} = \vec{a}_1 + \lambda\vec{b}_1$ and $\vec{r} = \vec{a}_2 + \mu\vec{b}_2$: $d = \left|\dfrac{(\vec{b}_1 \times \vec{b}_2)\cdot(\vec{a}_2 - \vec{a}_1)}{|\vec{b}_1 \times \vec{b}_2|}\right|$.

Plane

- Normal form: $lx + my + nz = d$ ($l, m, n =$ D.C.'s of normal, $d =$ distance from origin).
- Through a point $\vec{a}$, normal $\vec{N}$: $(\vec{r} - \vec{a}) \cdot \vec{N} = 0$.
- Through 3 points $\vec{a}, \vec{b}, \vec{c}$: $(\vec{r} - \vec{a}) \cdot [(\vec{b} - \vec{a}) \times (\vec{c} - \vec{a})] = 0$.
- Intercept form: $\dfrac{x}{p} + \dfrac{y}{q} + \dfrac{z}{r} = 1$.
- Angle between planes with normals $\vec{n}_1, \vec{n}_2$: $\cos\theta = \left|\dfrac{\vec{n}_1 \cdot \vec{n}_2}{|\vec{n}_1||\vec{n}_2|}\right|$.
- Distance of (x_1, y_1, z_1) from $Ax + By + Cz = D$: $\left|\dfrac{Ax_1 + By_1 + Cz_1 - D}{\sqrt{A^2+B^2+C^2}}\right|$.
- Angle between a line (D.R. a, b, c) and a plane (normal D.R. A, B, C): $\sin\phi = \left|\dfrac{Aa + Bb + Cc}{\sqrt{a^2+b^2+c^2}\sqrt{A^2+B^2+C^2}}\right|$.

1 Exercise 11.1 — Direction Cosines and Direction Ratios of a Line

Q1. If a line makes angles $90°, 135°, 45°$ with the x, y and z axes respectively, find its direction cosines.

The direction cosines of a line making angles α, β, γ with the coordinate axes are $\cos\alpha, \cos\beta, \cos\gamma$.

$$l = \cos 90° = 0, \qquad m = \cos 135° = -\frac{1}{\sqrt{2}}, \qquad n = \cos 45° = \frac{1}{\sqrt{2}}$$

Direction cosines $= \left(0,\ -\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}}\right)$.

Q2. Find the direction cosines of a line which makes equal angles with the coordinate axes.

Let the line make an angle α with each axis. Then $l = m = n = \cos\alpha$. Since $l^2 + m^2 + n^2 = 1$:

$$3\cos^2\alpha = 1 \ \Rightarrow\ \cos\alpha = \pm\frac{1}{\sqrt{3}}$$

Direction cosines $= \left(\pm\frac{1}{\sqrt{3}},\ \pm\frac{1}{\sqrt{3}},\ \pm\frac{1}{\sqrt{3}}\right)$ (all signs the same).

Q3. If a line has direction ratios $-18, 12, -4$, then what are its direction cosines?

Magnitude $= \sqrt{(-18)^2 + 12^2 + (-4)^2} = \sqrt{324 + 144 + 16} = \sqrt{484} = 22$.

$$l = \frac{-18}{22} = -\frac{9}{11}, \qquad m = \frac{12}{22} = \frac{6}{11}, \qquad n = \frac{-4}{22} = -\frac{2}{11}$$

Direction cosines $= \left(-\frac{9}{11},\ \frac{6}{11},\ -\frac{2}{11}\right)$.

Q4. Show that the points $(2, 3, 4)$, $(-1, -2, 1)$, $(5, 8, 7)$ are collinear.

Let $A(2, 3, 4)$, $B(-1, -2, 1)$, $C(5, 8, 7)$.
Direction ratios of AB: $(-1 - 2,\ -2 - 3,\ 1 - 4) = (-3, -5, -3)$.
Direction ratios of BC: $(5 - (-1),\ 8 - (-2),\ 7 - 1) = (6, 10, 6)$.
Since $(6, 10, 6) = -2 \times (-3, -5, -3)$, the direction ratios of AB and BC are proportional. As point B is common to both lines, A, B, C are **collinear**.

Q5. Find the direction cosines of the sides of the triangle whose vertices are $(3, 5, -4)$, $(-1, 1, 2)$ and $(-5, -5, -2)$.

Let $A(3, 5, -4)$, $B(-1, 1, 2)$, $C(-5, -5, -2)$.
Side AB**:** D.R.'s $= (-1-3,\ 1-5,\ 2-(-4)) = (-4, -4, 6)$. Magnitude $= \sqrt{16+16+36} = \sqrt{68} = 2\sqrt{17}$.

$$\text{D.C.'s of } AB = \left(\frac{-2}{\sqrt{17}},\ \frac{-2}{\sqrt{17}},\ \frac{3}{\sqrt{17}}\right)$$

Side BC**:** D.R.'s $= (-5-(-1),\ -5-1,\ -2-2) = (-4, -6, -4)$. Magnitude $= \sqrt{16+36+16} = \sqrt{68} = 2\sqrt{17}$.

$$\text{D.C.'s of } BC = \left(\frac{-2}{\sqrt{17}},\ \frac{-3}{\sqrt{17}},\ \frac{-2}{\sqrt{17}}\right)$$

Side CA**:** D.R.'s $= (3-(-5),\ 5-(-5),\ -4-(-2)) = (8, 10, -2)$. Magnitude $= \sqrt{64+100+4} = \sqrt{168} = 2\sqrt{42}$.

$$\text{D.C.'s of } CA = \left(\frac{4}{\sqrt{42}},\ \frac{5}{\sqrt{42}},\ \frac{-1}{\sqrt{42}}\right)$$

2 Exercise 11.2 — Equation of a Line in Space, Angle Between Lines, Shortest Distance

Q1. Show that the three lines with direction cosines $\frac{12}{13}, \frac{-3}{13}, \frac{-4}{13}$; $\frac{4}{13}, \frac{12}{13}, \frac{3}{13}$; $\frac{3}{13}, \frac{-4}{13}, \frac{12}{13}$ are mutually perpendicular.

Two lines with D.C.'s (l_1, m_1, n_1) and (l_2, m_2, n_2) are perpendicular if $l_1 l_2 + m_1 m_2 + n_1 n_2 = 0$.

$$L_1 \cdot L_2 = \frac{12 \times 4 + (-3)(12) + (-4)(3)}{169} = \frac{48 - 36 - 12}{169} = 0$$

$$L_2 \cdot L_3 = \frac{4 \times 3 + 12 \times (-4) + 3 \times 12}{169} = \frac{12 - 48 + 36}{169} = 0$$

$$L_1 \cdot L_3 = \frac{12 \times 3 + (-3)(-4) + (-4)(12)}{169} = \frac{36 + 12 - 48}{169} = 0$$

All three dot products are zero, so the three lines are **mutually perpendicular**.

Q2. Show that the line through the points $(1, -1, 2)$, $(3, 4, -2)$ is perpendicular to the line through the points $(0, 3, 2)$ and $(3, 5, 6)$.

D.R.'s of AB: $(3 - 1,\ 4 - (-1),\ -2 - 2) = (2, 5, -4)$.
D.R.'s of CD: $(3 - 0,\ 5 - 3,\ 6 - 2) = (3, 2, 4)$.

$$a_1 a_2 + b_1 b_2 + c_1 c_2 = 2(3) + 5(2) + (-4)(4) = 6 + 10 - 16 = 0$$

Hence $AB \perp CD$.

Q3. Show that the line through the points $(4, 7, 8)$, $(2, 3, 4)$ is parallel to the line through the points $(-1, -2, 1)$ and $(1, 2, 5)$.

D.R.'s of AB: $(2 - 4,\ 3 - 7,\ 4 - 8) = (-2, -4, -4)$, i.e. proportional to $(1, 2, 2)$.
D.R.'s of CD: $(1 - (-1),\ 2 - (-2),\ 5 - 1) = (2, 4, 4)$, i.e. proportional to $(1, 2, 2)$.
Since the direction ratios are proportional, $AB \parallel CD$.

Q4. Find the equation of the line which passes through the point $(1, 2, 3)$ and is parallel to the vector $3\hat{i} + 2\hat{j} - 2\hat{k}$.

Vector form:

$$\vec{r} = (\hat{i} + 2\hat{j} + 3\hat{k}) + \lambda(3\hat{i} + 2\hat{j} - 2\hat{k})$$

Cartesian form:

$$\frac{x - 1}{3} = \frac{y - 2}{2} = \frac{z - 3}{-2}$$

Q5. Find the equation of the line in vector and Cartesian form that passes through the point with position vector $2\hat{\imath} - \hat{\jmath} + 4\hat{k}$ and is in the direction $\hat{\imath} + 2\hat{\jmath} - \hat{k}$.

Vector form:

$$\vec{r} = (2\hat{\imath} - \hat{\jmath} + 4\hat{k}) + \lambda(\hat{\imath} + 2\hat{\jmath} - \hat{k})$$

Cartesian form:

$$\frac{x-2}{1} = \frac{y+1}{2} = \frac{z-4}{-1}$$

Q6. Find the Cartesian equation of the line which passes through the point $(-2, 4, -5)$ and is parallel to the line $\frac{x+3}{3} = \frac{y-4}{5} = \frac{z+8}{6}$.

The given line has direction ratios $(3, 5, 6)$. The required line, through $(-2, 4, -5)$ with the same direction ratios, is

$$\frac{x+2}{3} = \frac{y-4}{5} = \frac{z+5}{6}$$

Q7. The Cartesian equation of a line is $\frac{x-5}{3} = \frac{y+4}{7} = \frac{z-6}{2}$. Write its vector form.

The line passes through $(5, -4, 6)$ with direction ratios $(3, 7, 2)$.

$$\vec{r} = (5\hat{\imath} - 4\hat{\jmath} + 6\hat{k}) + \lambda(3\hat{\imath} + 7\hat{\jmath} + 2\hat{k})$$

Q8. Find the vector and Cartesian equations of the line that passes through the origin and $(5, -2, 3)$.

Direction ratios: $(5, -2, 3)$. **Vector form:**

$$\vec{r} = \lambda(5\hat{\imath} - 2\hat{\jmath} + 3\hat{k})$$

Cartesian form:

$$\frac{x}{5} = \frac{y}{-2} = \frac{z}{3}$$

Q9. Find the vector and Cartesian equations of the line that passes through the points $(3, -2, -5)$ and $(3, -2, 6)$.

Direction ratios: $(3 - 3,\ -2 - (-2),\ 6 - (-5)) = (0, 0, 11)$, i.e. proportional to $(0, 0, 1)$ — the line is parallel to the z-axis. **Vector form:**

$$\vec{r} = (3\hat{\imath} - 2\hat{\jmath} - 5\hat{k}) + \lambda\,\hat{k}$$

Cartesian form:

$$\frac{x-3}{0} = \frac{y+2}{0} = \frac{z+5}{1} \qquad (x = 3,\ y = -2,\ z \text{ variable})$$

Q10. Find the angle between the following pair of lines: $\vec{r} = 2\hat{i} - 5\hat{j} + \hat{k} + \lambda(3\hat{i} + 2\hat{j} + 6\hat{k})$ and $\vec{r} = 7\hat{i} - 6\hat{k} + \mu(\hat{i} + 2\hat{j} + 2\hat{k})$.

Direction vectors: $\vec{b}_1 = (3, 2, 6)$, $\vec{b}_2 = (1, 2, 2)$.

$$\vec{b}_1 \cdot \vec{b}_2 = 3 + 4 + 12 = 19, \qquad |\vec{b}_1| = \sqrt{9 + 4 + 36} = 7, \qquad |\vec{b}_2| = \sqrt{1 + 4 + 4} = 3$$

$$\cos\theta = \frac{19}{7 \times 3} = \frac{19}{21} \;\Rightarrow\; \theta = \cos^{-1}\left(\frac{19}{21}\right)$$

Q11. Find the angle between the following pairs of lines:
(i) $\frac{x-2}{2} = \frac{y-1}{5} = \frac{z+3}{-3}$ and $\frac{x+2}{-1} = \frac{y-4}{8} = \frac{z-5}{4}$
(ii) $\frac{x}{2} = \frac{y}{2} = \frac{z}{1}$ and $\frac{x-5}{4} = \frac{y-2}{1} = \frac{z-3}{8}$

(i) D.R.'s: $(2, 5, -3)$ and $(-1, 8, 4)$.

$$\text{Dot} = 2(-1) + 5(8) + (-3)(4) = -2 + 40 - 12 = 26$$

$$|\vec{v}_1| = \sqrt{4 + 25 + 9} = \sqrt{38}, \qquad |\vec{v}_2| = \sqrt{1 + 64 + 16} = 9$$

$$\cos\theta = \frac{26}{9\sqrt{38}} \;\Rightarrow\; \theta = \cos^{-1}\left(\frac{26}{9\sqrt{38}}\right)$$

(ii) D.R.'s: $(2, 2, 1)$ and $(4, 1, 8)$.

$$\text{Dot} = 8 + 2 + 8 = 18, \qquad |\vec{v}_1| = 3, \qquad |\vec{v}_2| = 9$$

$$\cos\theta = \frac{18}{27} = \frac{2}{3} \;\Rightarrow\; \theta = \cos^{-1}\left(\frac{2}{3}\right)$$

Q12. Find the value of p so that the lines $\frac{1-x}{3} = \frac{7y-14}{2p} = \frac{z-3}{2}$ and $\frac{7-7x}{3p} = \frac{y-5}{1} = \frac{6-z}{5}$ are at right angles.

Rewrite line 1: $\frac{x-1}{-3} = \frac{y-2}{2p/7} = \frac{z-3}{2}$, so D.R.'s are $\left(-3, \frac{2p}{7}, 2\right)$.
Rewrite line 2: $\frac{x-1}{-3p/7} = \frac{y-5}{1} = \frac{z-6}{-5}$, so D.R.'s are $\left(-\frac{3p}{7}, 1, -5\right)$.
For perpendicularity:

$$(-3)\left(-\frac{3p}{7}\right) + \left(\frac{2p}{7}\right)(1) + (2)(-5) = 0$$

$$\frac{9p}{7} + \frac{2p}{7} - 10 = 0 \;\Rightarrow\; \frac{11p}{7} = 10 \;\Rightarrow\; p = \frac{70}{11}$$

Q13. Show that the lines $\frac{x-5}{7} = \frac{y+2}{-5} = \frac{z}{1}$ and $\frac{x}{1} = \frac{y}{2} = \frac{z}{3}$ are perpendicular to each other.

D.R.'s: $(7,-5,1)$ and $(1,2,3)$.

$$7(1)+(-5)(2)+1(3)=7-10+3=0$$

Hence the two lines are **perpendicular**.

Q14. Find the shortest distance between the lines $\vec{r}=(\hat{i}+2\hat{j}+\hat{k})+\lambda(\hat{i}-\hat{j}+\hat{k})$ and $\vec{r}=(2\hat{i}-\hat{j}-\hat{k})+\mu(2\hat{i}+\hat{j}+2\hat{k})$.

Here $\vec{a}_1=(1,2,1)$, $\vec{b}_1=(1,-1,1)$, $\vec{a}_2=(2,-1,-1)$, $\vec{b}_2=(2,1,2)$.

$$\vec{a}_2-\vec{a}_1=(1,-3,-2)$$

$$\vec{b}_1\times\vec{b}_2=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & -1 & 1\\ 2 & 1 & 2\end{vmatrix}=\hat{i}(-2-1)-\hat{j}(2-2)+\hat{k}(1+2)=(-3,0,3)$$

$$(\vec{b}_1\times\vec{b}_2)\cdot(\vec{a}_2-\vec{a}_1)=(-3)(1)+0(-3)+3(-2)=-9$$

$$|\vec{b}_1\times\vec{b}_2|=\sqrt{9+0+9}=3\sqrt{2}$$

$$d=\frac{|-9|}{3\sqrt{2}}=\frac{3}{\sqrt{2}}=\frac{3\sqrt{2}}{2}\text{ units}$$

Q15. Find the shortest distance between the lines $\dfrac{x+1}{7}=\dfrac{y+1}{-6}=\dfrac{z+1}{1}$ and $\dfrac{x-3}{1}=\dfrac{y-5}{-2}=\dfrac{z-7}{1}$.

Here $\vec{a}_1=(-1,-1,-1)$, $\vec{b}_1=(7,-6,1)$, $\vec{a}_2=(3,5,7)$, $\vec{b}_2=(1,-2,1)$.

$$\vec{a}_2-\vec{a}_1=(4,6,8)$$

$$\vec{b}_1\times\vec{b}_2=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 7 & -6 & 1\\ 1 & -2 & 1\end{vmatrix}=\hat{i}(-6+2)-\hat{j}(7-1)+\hat{k}(-14+6)=(-4,-6,-8)$$

$$(\vec{b}_1\times\vec{b}_2)\cdot(\vec{a}_2-\vec{a}_1)=(-4)(4)+(-6)(6)+(-8)(8)=-116$$

$$|\vec{b}_1\times\vec{b}_2|=\sqrt{16+36+64}=\sqrt{116}=2\sqrt{29}$$

$$d=\frac{116}{2\sqrt{29}}=\frac{58}{\sqrt{29}}=2\sqrt{29}\text{ units}$$

Q16. Find the shortest distance between the lines $\vec{r}=(\hat{i}+2\hat{j}+3\hat{k})+\lambda(\hat{i}-3\hat{j}+2\hat{k})$ and $\vec{r}=(4\hat{i}+5\hat{j}+6\hat{k})+\mu(2\hat{i}+3\hat{j}+\hat{k})$.

Here $\vec{a}_1 = (1, 2, 3)$, $\vec{b}_1 = (1, -3, 2)$, $\vec{a}_2 = (4, 5, 6)$, $\vec{b}_2 = (2, 3, 1)$.

$$\vec{a}_2 - \vec{a}_1 = (3, 3, 3)$$

$$\vec{b}_1 \times \vec{b}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -3 & 2 \\ 2 & 3 & 1 \end{vmatrix} = \hat{i}(-3 - 6) - \hat{j}(1 - 4) + \hat{k}(3 + 6) = (-9, 3, 9)$$

$$(\vec{b}_1 \times \vec{b}_2) \cdot (\vec{a}_2 - \vec{a}_1) = (-9)(3) + 3(3) + 9(3) = 9$$

$$|\vec{b}_1 \times \vec{b}_2| = \sqrt{81 + 9 + 81} = \sqrt{171} = 3\sqrt{19}$$

$$d = \frac{9}{3\sqrt{19}} = \frac{3}{\sqrt{19}} = \frac{3\sqrt{19}}{19} \text{ units}$$

Q17. Find the shortest distance between the lines whose vector equations are $\vec{r} = (1 - t)\hat{i} + (t - 2)\hat{j} + (3 - 2t)\hat{k}$ and $\vec{r} = (s + 1)\hat{i} + (2s - 1)\hat{j} - (2s + 1)\hat{k}$.

Rewriting: Line 1: $\vec{a}_1 = (1, -2, 3)$, $\vec{b}_1 = (-1, 1, -2)$. Line 2: $\vec{a}_2 = (1, -1, -1)$, $\vec{b}_2 = (1, 2, -2)$.

$$\vec{a}_2 - \vec{a}_1 = (0, 1, -4)$$

$$\vec{b}_1 \times \vec{b}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & 1 & -2 \\ 1 & 2 & -2 \end{vmatrix} = \hat{i}(-2 + 4) - \hat{j}(2 + 2) + \hat{k}(-2 - 1) = (2, -4, -3)$$

$$(\vec{b}_1 \times \vec{b}_2) \cdot (\vec{a}_2 - \vec{a}_1) = 0 - 4 + 12 = 8$$

$$|\vec{b}_1 \times \vec{b}_2| = \sqrt{4 + 16 + 9} = \sqrt{29}$$

$$d = \frac{8}{\sqrt{29}} = \frac{8\sqrt{29}}{29} \text{ units}$$

3 Exercise 11.3 — The Plane

Q1. In each of the following cases, determine the direction cosines of the normal to the plane and the distance from the origin:
(a) $z = 2$ (b) $x + y + z = 1$ (c) $2x + 3y - z = 5$ (d) $5y + 8 = 0$

(a) $z = 2$, i.e. $0 \cdot x + 0 \cdot y + 1 \cdot z = 2$. Normal D.R.'s $(0, 0, 1)$, magnitude 1. D.C.'s $= (0, 0, 1)$, distance $= 2$.
(b) $x + y + z = 1$. Normal D.R.'s $(1, 1, 1)$, magnitude $\sqrt{3}$. D.C.'s $= \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$, distance $= \frac{1}{\sqrt{3}}$.
(c) $2x + 3y - z = 5$. Normal D.R.'s $(2, 3, -1)$, magnitude $\sqrt{14}$. D.C.'s $= \left(\frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}}, \frac{-1}{\sqrt{14}}\right)$, distance $= \frac{5}{\sqrt{14}}$.
(d) $5y + 8 = 0 \Rightarrow 0x - 5y + 0z = 8$. Normal D.R.'s $(0, -5, 0)$, magnitude 5. D.C.'s $= (0, -1, 0)$, distance $= \frac{8}{5}$.

Q2. Find the vector equation of a plane which is at a distance of 7 units from the origin and normal to the vector $3\hat{i} + 5\hat{j} - 6\hat{k}$.

Unit normal:

$$\hat{n} = \frac{3\hat{i} + 5\hat{j} - 6\hat{k}}{\sqrt{9 + 25 + 36}} = \frac{3\hat{i} + 5\hat{j} - 6\hat{k}}{\sqrt{70}}$$

Vector equation ($\vec{r} \cdot \hat{n} = d$):

$$\vec{r} \cdot \left(\frac{3\hat{i} + 5\hat{j} - 6\hat{k}}{\sqrt{70}}\right) = 7$$

Q3. Find the Cartesian equation of the following planes:
(a) $\vec{r} \cdot (\hat{i} + \hat{j} - \hat{k}) = 2$ (b) $\vec{r} \cdot (2\hat{i} + 3\hat{j} - 4\hat{k}) = 1$ (c) $\vec{r} \cdot [(s - 2t)\hat{i} + (3 - t)\hat{j} + (2s + t)\hat{k}] = 15$

Let $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$.
(a) $(x\hat{i} + y\hat{j} + z\hat{k}) \cdot (\hat{i} + \hat{j} - \hat{k}) = 2 \Rightarrow x + y - z = 2$.
(b) $(x\hat{i} + y\hat{j} + z\hat{k}) \cdot (2\hat{i} + 3\hat{j} - 4\hat{k}) = 1 \Rightarrow 2x + 3y - 4z = 1$.
(c) $(x\hat{i} + y\hat{j} + z\hat{k}) \cdot [(s - 2t)\hat{i} + (3 - t)\hat{j} + (2s + t)\hat{k}] = 15$

$$\Rightarrow (s - 2t)\,x + (3 - t)\,y + (2s + t)\,z = 15$$

Q4. In the following cases, find the coordinates of the foot of the perpendicular drawn from the origin:
(a) $2x + 3y + 4z - 12 = 0$ (b) $3y + 4z - 6 = 0$ (c) $x + y + z = 1$ (d) $5y + 8 = 0$

(a) Normal D.R.'s $(2,3,4)$, magnitude $\sqrt{29}$. Foot of perpendicular $= \frac{12}{29}(2,3,4) = \left(\frac{24}{29}, \frac{36}{29}, \frac{48}{29}\right)$.
(b) Normal D.R.'s $(0,3,4)$, magnitude 5. Foot $= \frac{6}{25}(0,3,4) = \left(0, \frac{18}{25}, \frac{24}{25}\right)$.
(c) Normal D.R.'s $(1,1,1)$, magnitude $\sqrt{3}$. Foot $= \frac{1}{3}(1,1,1) = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$.
(d) Normal D.R.'s $(0,-5,0)$, magnitude 5. Foot $= \frac{8}{25}(0,-5,0) = \left(0, -\frac{8}{5}, 0\right)$.

Q5. Find the vector and Cartesian equations of the planes:
(a) that passes through the point $(1,0,-2)$ and the normal to the plane is $\hat{i}+\hat{j}-\hat{k}$
(b) that passes through the point $(1,4,6)$ and the normal vector to the plane is $\hat{i}-2\hat{j}+\hat{k}$

(a) $\vec{a}=\hat{i}-2\hat{k}$, $\vec{N}=\hat{i}+\hat{j}-\hat{k}$. Vector eq.: $(\vec{r}-\vec{a})\cdot\vec{N}=0$.

$$(x-1)(1)+y(1)+(z+2)(-1)=0 \;\Rightarrow\; x+y-z=3$$

(b) $\vec{a}=\hat{i}+4\hat{j}+6\hat{k}$, $\vec{N}=\hat{i}-2\hat{j}+\hat{k}$.

$$(x-1)(1)+(y-4)(-2)+(z-6)(1)=0 \;\Rightarrow\; x-2y+z=-1$$

Q6. Find the equations of the planes that pass through three points:
(a) $(1,1,-1)$, $(6,4,-5)$, $(-4,-2,3)$ (b) $(1,1,0)$, $(1,2,1)$, $(-2,2,-1)$

(a) Let plane through $(1,1,-1)$ be $a(x-1)+b(y-1)+c(z+1)=0$. Through $(6,4,-5)$: $5a+3b-4c=0$. Through $(-4,-2,3)$: $-5a-3b+4c=0$, which is the *same* equation as above. So $a:b:c$ is not uniquely determined — in fact the three given points are **collinear** (checking: $(-4,-2,3)-(1,1,-1)=(-5,-3,4)=-1\times(5,3,-4)$, i.e. $(6,4,-5)-(1,1,-1)$ negated). Since infinitely many planes pass through a straight line, **no unique plane** exists through these three points.
(b) Let plane through $(1,1,0)$ be $a(x-1)+b(y-1)+c\,z=0$. Through $(1,2,1)$: $0a+1b+1c=0 \Rightarrow b+c=0$. Through $(-2,2,-1)$: $-3a+1b-1c=0 \Rightarrow -3a+b-c=0$. From $b=-c$: substitute into second equation: $-3a-c-c=0 \Rightarrow -3a-2c=0 \Rightarrow a=-\frac{2c}{3}$. Taking $c=-3$: $a=2,\ b=3,\ c=-3$.

$$2(x-1)+3(y-1)-3z=0 \;\Rightarrow\; 2x+3y-3z=5$$

Q7. Find the intercepts cut off by the plane $2x+y-z=5$.

Divide by 5:

$$\frac{x}{5/2}+\frac{y}{5}+\frac{z}{-5}=1$$

Intercepts: on x-axis $=\frac{5}{2}$, on y-axis $=5$, on z-axis $=-5$.

Q8. Find the equation of the plane with intercept 3 on the y-axis and parallel to the ZOX plane.

Any plane parallel to the ZOX plane has the form $y = b$. Since the intercept on the y-axis is 3,

$$y = 3$$

Q9. Find the equation of the plane through the intersection of the planes $3x - y + 2z - 4 = 0$ and $x + y + z - 2 = 0$ and the point $(2, 2, 1)$.

Family of planes: $(3x - y + 2z - 4) + \lambda(x + y + z - 2) = 0$. Substituting $(2, 2, 1)$: $(6 - 2 + 2 - 4) + \lambda(2 + 2 + 1 - 2) = 0 \Rightarrow 2 + 3\lambda = 0 \Rightarrow \lambda = -\frac{2}{3}$.

$$(3x - y + 2z - 4) - \frac{2}{3}(x + y + z - 2) = 0$$

Multiplying by 3: $9x - 3y + 6z - 12 - 2x - 2y - 2z + 4 = 0$

$$\Rightarrow \; 7x - 5y + 4z - 8 = 0$$

Q10. Find the vector equation of the plane passing through the intersection of the planes $\vec{r} \cdot (2\hat{i} + 2\hat{j} - 3\hat{k}) = 7$, $\vec{r} \cdot (2\hat{i} + 5\hat{j} + 3\hat{k}) = 9$ and through the point $(2, 1, 3)$.

Family: $(2x + 2y - 3z - 7) + \lambda(2x + 5y + 3z - 9) = 0$. At $(2, 1, 3)$: $(4 + 2 - 9 - 7) + \lambda(4 + 5 + 9 - 9) = 0 \Rightarrow -10 + 9\lambda = 0 \Rightarrow \lambda = \frac{10}{9}$.

$$(2x + 2y - 3z - 7) + \frac{10}{9}(2x + 5y + 3z - 9) = 0$$

Multiplying by 9: $18x + 18y - 27z - 63 + 20x + 50y + 30z - 90 = 0$

$$\Rightarrow \; 38x + 68y + 3z = 153$$

Vector form: $\vec{r} \cdot (38\hat{i} + 68\hat{j} + 3\hat{k}) = 153$.

Q11. Find the equation of the plane through the line of intersection of the planes $x + y + z = 1$ and $2x + 3y + 4z = 5$ which is perpendicular to the plane $x - y + z = 0$.

Family: $(x + y + z - 1) + \lambda(2x + 3y + 4z - 5) = 0$

$$\Rightarrow \; (2\lambda + 1)x + (3\lambda + 1)y + (4\lambda + 1)z - (5\lambda + 1) = 0$$

This must be perpendicular to $x - y + z = 0$ (normal $(1, -1, 1)$):

$$(2\lambda + 1)(1) + (3\lambda + 1)(-1) + (4\lambda + 1)(1) = 0$$

$$2\lambda + 1 - 3\lambda - 1 + 4\lambda + 1 = 0 \; \Rightarrow \; 3\lambda + 1 = 0 \; \Rightarrow \; \lambda = -\frac{1}{3}$$

Substituting back:

$$\left(1-\frac{2}{3}\right)x+(1-1)\,y+\left(1-\frac{4}{3}\right)z-\left(1-\frac{5}{3}\right)=0$$

$$\frac{1}{3}x+0y-\frac{1}{3}z+\frac{2}{3}=0$$

Multiplying by 3:

$$x-z+2=0$$

Q12. Find the angle between the planes whose vector equations are $\vec{r}\cdot(2\hat{\imath}+2\hat{\jmath}-3\hat{k})=5$ and $\vec{r}\cdot(3\hat{\imath}-3\hat{\jmath}+5\hat{k})=3$.

Normals: $\vec{n}_1=(2,2,-3)$, $\vec{n}_2=(3,-3,5)$.

$$\vec{n}_1\cdot\vec{n}_2=6-6-15=-15$$

$$|\vec{n}_1|=\sqrt{4+4+9}=\sqrt{17},\qquad |\vec{n}_2|=\sqrt{9+9+25}=\sqrt{43}$$

$$\cos\theta=\left|\frac{-15}{\sqrt{17}\sqrt{43}}\right|=\frac{15}{\sqrt{731}}\ \Rightarrow\ \theta=\cos^{-1}\left(\frac{15}{\sqrt{731}}\right)$$

Q13. In the following cases, determine whether the given planes are parallel or perpendicular, and in case they are neither, find the angle between them:
(a) $7x+5y+6z+30=0$ and $3x-y-10z+4=0$
(b) $2x+y+3z-2=0$ and $x-2y+5=0$
(c) $2x-2y+4z+5=0$ and $3x-3y+6z-1=0$
(d) $2x-y+3z-1=0$ and $2x-y+3z+3=0$
(e) $4x+8y+z-8=0$ and $y+z-4=0$

(a) Normals $(7,5,6)$, $(3,-1,-10)$. Dot $=21-5-60=-44\neq 0$; ratios $\frac{7}{3},\frac{-5}{1},\frac{6}{-10}$ unequal $\Rightarrow$ **neither**.

$$|\vec{n}_1|=\sqrt{49+25+36}=\sqrt{110},\quad |\vec{n}_2|=\sqrt{9+1+100}=\sqrt{110}$$

$$\cos\theta=\left|\frac{-44}{110}\right|=\frac{2}{5}\ \Rightarrow\ \theta=\cos^{-1}\left(\frac{2}{5}\right)$$

(b) Normals $(2,1,3)$, $(1,-2,0)$. Dot $=2-2+0=0\Rightarrow$ **perpendicular**.
(c) Normals $(2,-2,4)$, $(3,-3,6)$. Since $(3,-3,6)=\frac{3}{2}(2,-2,4)$, ratios are equal $\Rightarrow$ **parallel**.
(d) Normals $(2,-1,3)$, $(2,-1,3)$ — identical direction $\Rightarrow$ **parallel** (these are, in fact, parallel distinct planes).
(e) Normals $(4,8,1)$, $(0,1,1)$. Dot $=0+8+1=9\neq 0$; not parallel $\Rightarrow$ **neither**.

$$|\vec{n}_1|=\sqrt{16+64+1}=9,\quad |\vec{n}_2|=\sqrt{0+1+1}=\sqrt{2}$$

$$\cos\theta=\frac{9}{9\sqrt{2}}=\frac{1}{\sqrt{2}}\ \Rightarrow\ \theta=45^\circ$$

Q14. In the following cases, find the distance of each of the given points from the corresponding given plane:

Point	Plane
(a) $(0, 0, 0)$	$3x - 4y + 12z = 3$
(b) $(3, -2, 1)$	$2x - y + 2z + 3 = 0$
(c) $(2, 3, -5)$	$x + 2y - 2z = 9$
(d) $(-6, 0, 0)$	$2x - 3y + 6z - 2 = 0$

Distance of (x_1, y_1, z_1) from $Ax + By + Cz + D = 0$ is $\left|\dfrac{Ax_1 + By_1 + Cz_1 + D}{\sqrt{A^2 + B^2 + C^2}}\right|$.

(a) Plane: $3x - 4y + 12z - 3 = 0$. Distance $= \left|\dfrac{0 - 0 + 0 - 3}{\sqrt{9 + 16 + 144}}\right| = \dfrac{3}{13}$.

(b) Distance $= \left|\dfrac{2(3) - (-2) + 2(1) + 3}{\sqrt{4 + 1 + 4}}\right| = \left|\dfrac{6 + 2 + 2 + 3}{3}\right| = \dfrac{13}{3}$.

(c) Plane: $x + 2y - 2z - 9 = 0$. Distance $= \left|\dfrac{2 + 6 + 10 - 9}{\sqrt{1 + 4 + 4}}\right| = \dfrac{9}{3} = 3$.

(d) Distance $= \left|\dfrac{2(-6) - 0 + 0 - 2}{\sqrt{4 + 9 + 36}}\right| = \left|\dfrac{-14}{7}\right| = 2$.

Q15. Find the angle between the line $\dfrac{x+1}{2} = \dfrac{y}{3} = \dfrac{z-3}{6}$ and the plane $10x + 2y - 11z = 3$.

Line D.R.'s: $(2, 3, 6)$. Plane normal D.R.'s: $(10, 2, -11)$.

$$\sin\phi = \left|\frac{Aa + Bb + Cc}{\sqrt{a^2 + b^2 + c^2}\sqrt{A^2 + B^2 + C^2}}\right| = \left|\frac{10(2) + 2(3) + (-11)(6)}{\sqrt{4 + 9 + 36}\sqrt{100 + 4 + 121}}\right|$$

$$= \left|\frac{20 + 6 - 66}{\sqrt{49}\sqrt{225}}\right| = \left|\frac{-40}{7 \times 15}\right| = \frac{8}{21}$$

$$\phi = \sin^{-1}\left(\frac{8}{21}\right)$$

4 Miscellaneous Exercise — Selected Board-Exam Style Problems

The Miscellaneous Exercise mixes ideas from the whole chapter. The problems below are a representative selection of the most frequently asked and board-exam-relevant questions from this exercise, each solved completely.

Q1. Show that the line joining the origin to the point $(2, 1, 1)$ is perpendicular to the line determined by the points $(3, 5, -1)$ and $(4, 3, -1)$.

D.R.'s of OA (through $O(0, 0, 0)$ and $A(2, 1, 1)$): $(2, 1, 1)$. D.R.'s of BC (through $B(3, 5, -1)$, $C(4, 3, -1)$): $(1, -2, 0)$.

$$2(1) + 1(-2) + 1(0) = 2 - 2 = 0$$

Hence $OA \perp BC$.

Q2. If l_1, m_1, n_1 and l_2, m_2, n_2 are the direction cosines of two mutually perpendicular lines, show that the direction cosines of the line perpendicular to both are

$$m_1n_2 - m_2n_1, \quad n_1l_2 - n_2l_1, \quad l_1m_2 - l_2m_1.$$

Since the two given lines are perpendicular: $l_1l_2 + m_1m_2 + n_1n_2 = 0$, and each is a set of D.C.'s, so $l_1^2 + m_1^2 + n_1^2 = 1$ and $l_2^2 + m_2^2 + n_2^2 = 1$.
Using the vector identity (Lagrange's identity)

$$(l_1^2+m_1^2+n_1^2)(l_2^2+m_2^2+n_2^2)-(l_1l_2+m_1m_2+n_1n_2)^2 = (m_1n_2-m_2n_1)^2+(n_1l_2-n_2l_1)^2+(l_1m_2-l_2m_1)^2$$

Substituting the known values, the left side becomes $1 \times 1 - 0^2 = 1$. So

$$(m_1n_2 - m_2n_1)^2 + (n_1l_2 - n_2l_1)^2 + (l_1m_2 - l_2m_1)^2 = 1$$

This shows $m_1n_2 - m_2n_1,\ n_1l_2 - n_2l_1,\ l_1m_2 - l_2m_1$ satisfy the defining property of direction cosines (sum of squares $= 1$); and since this vector is $\vec{b}_1 \times \vec{b}_2$ (up to the D.C. vectors of the two lines), it is perpendicular to both given lines. Hence these are the direction cosines of the line perpendicular to both.

Q3. Find the angle between the lines whose direction ratios are a, b, c and $b - c,\ c - a,\ a - b$.

$$\cos\theta = \left|\frac{a(b-c) + b(c-a) + c(a-b)}{\sqrt{a^2+b^2+c^2}\sqrt{(b-c)^2+(c-a)^2+(a-b)^2}}\right|$$

Numerator: $ab - ac + bc - ab + ca - cb = 0$.

$$\cos\theta = 0 \ \Rightarrow\ \theta = 90^\circ$$

Q4. Find the equation of a line parallel to the x-axis and passing through the origin.

A line parallel to the x-axis and through the origin is the x-axis itself, with direction ratios $(1, 0, 0)$.

$$\frac{x}{1} = \frac{y}{0} = \frac{z}{0}$$

Q5. If the coordinates of the points A, B, C, D are $(1, 2, 3)$, $(4, 5, 7)$, $(-4, 3, -6)$ and $(2, 9, 2)$ respectively, find the angle between the lines AB and CD.

D.R.'s of AB: $(4 - 1,\ 5 - 2,\ 7 - 3) = (3, 3, 4)$.
D.R.'s of CD: $(2 - (-4),\ 9 - 3,\ 2 - (-6)) = (6, 6, 8)$.
Since $(6, 6, 8) = 2(3, 3, 4)$, the direction ratios are proportional, so $AB \parallel CD$, and the angle between them is $0°$.

Q6. If the lines $\frac{x-1}{-3} = \frac{y-2}{2k} = \frac{z-3}{2}$ and $\frac{x-1}{3k} = \frac{y-1}{1} = \frac{z-6}{-5}$ are perpendicular, find the value of k.

D.R.'s: $(-3, 2k, 2)$ and $(3k, 1, -5)$.

$$(-3)(3k) + (2k)(1) + (2)(-5) = 0$$

$$-9k + 2k - 10 = 0 \;\Rightarrow\; -7k = 10 \;\Rightarrow\; k = -\frac{10}{7}$$

Q7. Find the vector equation of the line passing through $(1, 2, 3)$ and perpendicular to the plane $\vec{r} \cdot (\hat{i} + 2\hat{j} - 5\hat{k}) + 9 = 0$.

The line is parallel to the normal of the plane, $\hat{i} + 2\hat{j} - 5\hat{k}$.

$$\vec{r} = (\hat{i} + 2\hat{j} + 3\hat{k}) + \lambda(\hat{i} + 2\hat{j} - 5\hat{k})$$

Q8. Find the equation of the plane passing through (a, b, c) and parallel to the plane $\vec{r} \cdot (\hat{i} + \hat{j} + \hat{k}) = 2$.

Any plane parallel to $x + y + z = 2$ has the form $x + y + z = k$. Since it passes through (a, b, c): $k = a + b + c$.

$$x + y + z = a + b + c \qquad \text{i.e.} \qquad \vec{r} \cdot (\hat{i} + \hat{j} + \hat{k}) = a + b + c$$

Q9. Find the shortest distance between the lines $\vec{r} = 6\hat{i} + 2\hat{j} + 2\hat{k} + \lambda(\hat{i} - 2\hat{j} + 2\hat{k})$ and $\vec{r} = -4\hat{i} - \hat{k} + \mu(3\hat{i} - 2\hat{j} - 2\hat{k})$.

$\vec{a}_1 = (6, 2, 2)$, $\vec{b}_1 = (1, -2, 2)$, $\vec{a}_2 = (-4, 0, -1)$, $\vec{b}_2 = (3, -2, -2)$.

$$\vec{a}_2 - \vec{a}_1 = (-10, -2, -3)$$

$$\vec{b}_1 \times \vec{b}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -2 & 2 \\ 3 & -2 & -2 \end{vmatrix} = \hat{i}(4+4) - \hat{j}(-2-6) + \hat{k}(-2+6) = (8, 8, 4)$$

$$(\vec{b}_1 \times \vec{b}_2) \cdot (\vec{a}_2 - \vec{a}_1) = 8(-10) + 8(-2) + 4(-3) = -108$$

$$|\vec{b}_1 \times \vec{b}_2| = \sqrt{64 + 64 + 16} = \sqrt{144} = 12$$

$$d = \frac{108}{12} = 9 \text{ units}$$

Chapter Summary Checklist

- ☐ Convert between angles-with-axes and direction cosines; know $l^2 + m^2 + n^2 = 1$.
- ☐ Find direction ratios between two points, and normalise them to direction cosines.
- ☐ Write the vector and Cartesian form of a line through a point (or through two points).
- ☐ Use the dot product to test perpendicularity $(= 0)$ and proportional ratios to test parallelism.
- ☐ Compute the angle between two lines using direction ratios or vectors.
- ☐ Apply the cross-product formula for the shortest distance between skew lines.
- ☐ Write a plane's equation from a point + normal, from three points, in intercept form, or through the intersection of two planes.
- ☐ Find the foot of the perpendicular from the origin, and the distance of any point from a plane.
- ☐ Determine whether two planes are parallel, perpendicular, or find the angle between them.
- ☐ Find the angle between a line and a plane using the $\sin\phi$ formula.

End of Chapter 11 — Three Dimensional Geometry
Practice each solved question by hand before checking the solution, and always double-check your final answer's units and sign.