NCERT Class 12 Maths

Chapter-wise Complete Solutions

Chapter 2

Inverse Trigonometric Functions

Complete Step-by-Step NCERT Solutions

Exercise 2.1 • Exercise 2.2 • Miscellaneous Exercise

What's inside:

- Every question solved with full, exam-ready working
- Key formulas & properties recap before each exercise
- Clear mathematical notation throughout
- Important tips on principal-value ranges (common mistake area)
- A quick-revision formula sheet at the end

Prepared for Class 12 CBSE / NCERT students

Suitable for board-exam preparation and quick revision

Contents

Chapter Overview & Key Formulas

Inverse trigonometric functions are the *inverse relations* of the basic trigonometric functions. Since trigonometric functions are not one-one over their entire domain, we restrict them to a specific interval called the **principal value branch** to make them invertible. Keep the table below handy — nearly every mistake in this chapter comes from using the wrong principal-value range.

Domain and Range (Principal Value Branch)

Function	**Domain**	**Range (Principal Branch)**
$\sin^{-1} x$	$[-1, 1]$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
$\cos^{-1} x$	$[-1, 1]$	$[0, \pi]$
$\tan^{-1} x$	$\mathbb{R}$	$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$
$\cot^{-1} x$	$\mathbb{R}$	$(0, \pi)$
$\sec^{-1} x$	$\mathbb{R} - (-1, 1)$	$[0, \pi] - \left\{\frac{\pi}{2}\right\}$
$\text{cosec}^{-1} x$	$\mathbb{R} - (-1, 1)$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\}$

Important Properties

Reciprocal relations:

$$\sin^{-1}\left(\frac{1}{x}\right) = \text{cosec}^{-1} x, \quad \cos^{-1}\left(\frac{1}{x}\right) = \sec^{-1} x, \quad \tan^{-1}\left(\frac{1}{x}\right) = \cot^{-1} x \ (x > 0)$$

Odd functions:

$$\sin^{-1}(-x) = -\sin^{-1} x, \quad \tan^{-1}(-x) = -\tan^{-1} x, \quad \text{cosec}^{-1}(-x) = -\text{cosec}^{-1} x$$

Supplementary-type relations:

$$\cos^{-1}(-x) = \pi - \cos^{-1} x, \quad \cot^{-1}(-x) = \pi - \cot^{-1} x, \quad \sec^{-1}(-x) = \pi - \sec^{-1} x$$

Complementary relations:

$$\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}, \quad \tan^{-1} x + \cot^{-1} x = \frac{\pi}{2}, \quad \sec^{-1} x + \text{cosec}^{-1} x = \frac{\pi}{2}$$

Sum/difference formulas:

$$\tan^{-1} x + \tan^{-1} y = \tan^{-1}\left(\frac{x + y}{1 - xy}\right), \ xy < 1$$

$$\tan^{-1} x - \tan^{-1} y = \tan^{-1}\left(\frac{x - y}{1 + xy}\right), \ xy > -1$$

Double-angle type:

$$2\tan^{-1} x = \sin^{-1}\left(\frac{2x}{1 + x^2}\right) = \cos^{-1}\left(\frac{1 - x^2}{1 + x^2}\right) = \tan^{-1}\left(\frac{2x}{1 - x^2}\right)$$

Tip: Whenever an angle falls outside a function's principal branch, first rewrite it using an appropriate identity (e.g. $\sin(\pi - \theta) = \sin\theta$, $\tan(\pi + \theta) = \tan\theta$) so it lands back inside the correct range — *then* apply the inverse function.

1 Exercise 2.1

Find the principal value of the following (Q1–Q10), and evaluate the given expressions (Q11–Q14).

Question 1: Find the principal value of $\sin^{-1}\left(-\frac{1}{2}\right)$.

Solution: Let $\sin^{-1}\left(-\frac{1}{2}\right) = y$. Then $\sin y = -\frac{1}{2} = -\sin\frac{\pi}{6} = \sin\left(-\frac{\pi}{6}\right)$.

Since the range of the principal branch of $\sin^{-1}$ is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ and $-\frac{\pi}{6}$ lies in this range,

$$\sin^{-1}\left(-\frac{1}{2}\right) = -\frac{\pi}{6}$$

Question 2: Find the principal value of $\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$.

Solution: Let $\cos^{-1}\left(\frac{\sqrt{3}}{2}\right) = y$. Then $\cos y = \frac{\sqrt{3}}{2} = \cos\frac{\pi}{6}$.

Since range of $\cos^{-1}$ is $[0, \pi]$ and $\frac{\pi}{6} \in [0, \pi]$,

$$\cos^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{6}$$

Question 3: Find the principal value of $\operatorname{cosec}^{-1}(2)$.

Solution: Let $\operatorname{cosec}^{-1}(2) = y$. Then $\operatorname{cosec} y = 2 = \operatorname{cosec}\frac{\pi}{6}$.

Since the range of $\operatorname{cosec}^{-1}$ is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\}$ and $\frac{\pi}{6}$ lies in this range,

$$\operatorname{cosec}^{-1}(2) = \frac{\pi}{6}$$

Question 4: Find the principal value of $\tan^{-1}(-\sqrt{3})$.

Solution: Let $\tan^{-1}(-\sqrt{3}) = y$. Then $\tan y = -\sqrt{3} = -\tan\frac{\pi}{3} = \tan\left(-\frac{\pi}{3}\right)$.

Since range of $\tan^{-1}$ is $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and $-\frac{\pi}{3}$ lies in it,

$$\tan^{-1}(-\sqrt{3}) = -\frac{\pi}{3}$$

Question 5: Find the principal value of $\cos^{-1}\left(-\frac{1}{2}\right)$.

Solution: Let $\cos^{-1}\left(-\frac{1}{2}\right) = y$. Then $\cos y = -\frac{1}{2} = -\cos\frac{\pi}{3} = \cos\left(\pi - \frac{\pi}{3}\right) = \cos\frac{2\pi}{3}$.

Since range of $\cos^{-1}$ is $[0, \pi]$ and $\frac{2\pi}{3} \in [0, \pi]$,

$$\cos^{-1}\left(-\frac{1}{2}\right) = \frac{2\pi}{3}$$

Question 6: Find the principal value of $\tan^{-1}(-1)$.

Solution: Let $\tan^{-1}(-1) = y$. Then $\tan y = -1 = -\tan\frac{\pi}{4} = \tan\left(-\frac{\pi}{4}\right)$.

Since $-\frac{\pi}{4} \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$,

$$\tan^{-1}(-1) = -\frac{\pi}{4}$$

Question 7: Find the principal value of $\sec^{-1}\left(\frac{2}{\sqrt{3}}\right)$.

Solution: Let $\sec^{-1}\left(\frac{2}{\sqrt{3}}\right) = y$. Then $\sec y = \frac{2}{\sqrt{3}} = \sec\frac{\pi}{6}$.

Since range of $\sec^{-1}$ is $[0, \pi] - \left\{\frac{\pi}{2}\right\}$ and $\frac{\pi}{6}$ lies in it,

$$\sec^{-1}\left(\frac{2}{\sqrt{3}}\right) = \frac{\pi}{6}$$

Question 8: Find the principal value of $\cot^{-1}(\sqrt{3})$.

Solution: Let $\cot^{-1}(\sqrt{3}) = y$. Then $\cot y = \sqrt{3} = \cot\dfrac{\pi}{6}$.

Since range of $\cot^{-1}$ is $(0, \pi)$ and $\dfrac{\pi}{6} \in (0, \pi)$,

$$\cot^{-1}(\sqrt{3}) = \frac{\pi}{6}$$

Question 9: Find the principal value of $\cos^{-1}\left(-\dfrac{1}{\sqrt{2}}\right)$.

Solution: Let $\cos^{-1}\left(-\dfrac{1}{\sqrt{2}}\right) = y$. Then $\cos y = -\dfrac{1}{\sqrt{2}} = -\cos\dfrac{\pi}{4} = \cos\left(\pi - \dfrac{\pi}{4}\right) = \cos\dfrac{3\pi}{4}$.

Since $\dfrac{3\pi}{4} \in [0, \pi]$,

$$\cos^{-1}\left(-\frac{1}{\sqrt{2}}\right) = \frac{3\pi}{4}$$

Question 10: Find the principal value of $\operatorname{cosec}^{-1}(-\sqrt{2})$.

Solution: Let $\operatorname{cosec}^{-1}(-\sqrt{2}) = y$. Then $\operatorname{cosec} y = -\sqrt{2} = -\operatorname{cosec}\dfrac{\pi}{4} = \operatorname{cosec}\left(-\dfrac{\pi}{4}\right)$.

Since $-\dfrac{\pi}{4} \in \left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right] - \{0\}$,

$$\operatorname{cosec}^{-1}(-\sqrt{2}) = -\frac{\pi}{4}$$

Question 11: Find the value of $\tan^{-1}(1) + \cos^{-1}\left(-\dfrac{1}{2}\right) + \sin^{-1}\left(-\dfrac{1}{2}\right)$.

Solution: We know:

$$\tan^{-1}(1) = \frac{\pi}{4}, \qquad \cos^{-1}\left(-\frac{1}{2}\right) = \frac{2\pi}{3}, \qquad \sin^{-1}\left(-\frac{1}{2}\right) = -\frac{\pi}{6}$$

Adding,

$$\frac{\pi}{4} + \frac{2\pi}{3} - \frac{\pi}{6} = \frac{3\pi + 8\pi - 2\pi}{12} = \frac{9\pi}{12} = \frac{3\pi}{4}$$

$$\boxed{\tan^{-1}(1) + \cos^{-1}\left(-\frac{1}{2}\right) + \sin^{-1}\left(-\frac{1}{2}\right) = \frac{3\pi}{4}}$$

Question 12: Find the value of $\cos^{-1}\left(\frac{1}{2}\right) + 2\sin^{-1}\left(\frac{1}{2}\right)$.

Solution: We know:

$$\cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3}, \qquad \sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}$$

So,

$$\cos^{-1}\left(\frac{1}{2}\right) + 2\sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3} + 2 \times \frac{\pi}{6} = \frac{\pi}{3} + \frac{\pi}{3} = \frac{2\pi}{3}$$

$$\boxed{\cos^{-1}\left(\frac{1}{2}\right) + 2\sin^{-1}\left(\frac{1}{2}\right) = \frac{2\pi}{3}}$$

Question 13: If $\sin^{-1} x = y$, then
(A) $0 \le y \le \pi$ (B) $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$ (C) $0 < y < \pi$ (D) $-\frac{\pi}{2} < y < \frac{\pi}{2}$

Solution: By definition, the principal value branch of $\sin^{-1} x$ has range $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

Hence if $\sin^{-1} x = y$, then $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$.

$$\boxed{\text{Option (B) is correct}}$$

Question 14: $\tan^{-1}\sqrt{3} - \sec^{-1}(-2)$ is equal to
(A) π (B) $-\frac{\pi}{3}$ (C) $\frac{\pi}{3}$ (D) $\frac{2\pi}{3}$

Solution:

$$\tan^{-1}\sqrt{3} = \frac{\pi}{3}$$

For $\sec^{-1}(-2)$: we need $y \in [0, \pi] - \left\{\frac{\pi}{2}\right\}$ with $\sec y = -2$.

$$\sec y = -2 = -\sec\frac{\pi}{3} = \sec\left(\pi - \frac{\pi}{3}\right) = \sec\frac{2\pi}{3} \;\Rightarrow\; \sec^{-1}(-2) = \frac{2\pi}{3}$$

Therefore,

$$\tan^{-1}\sqrt{3} - \sec^{-1}(-2) = \frac{\pi}{3} - \frac{2\pi}{3} = -\frac{\pi}{3}$$

$$\boxed{\text{Option (B) is correct}}$$

2 Exercise 2.2

Write functions in the simplest form (Q1–Q4) and prove the given identities (Q5–Q12) using the properties of inverse trigonometric functions.

Question 1: Write in the simplest form: $\tan^{-1}\left(\frac{\sqrt{1-\cos x}}{\sqrt{1+\cos x}}\right)$, $0 < x < \pi$.

Solution: Using $1-\cos x = 2\sin^2\frac{x}{2}$ and $1+\cos x = 2\cos^2\frac{x}{2}$:

$$\sqrt{\frac{1-\cos x}{1+\cos x}} = \sqrt{\frac{2\sin^2\frac{x}{2}}{2\cos^2\frac{x}{2}}} = \left|\tan\frac{x}{2}\right|$$

Since $0 < x < \pi \Rightarrow 0 < \frac{x}{2} < \frac{\pi}{2}$, $\tan\frac{x}{2} > 0$, so the modulus can be dropped:

$$\tan^{-1}\left(\tan\frac{x}{2}\right) = \frac{x}{2} \quad \left(\text{since } \frac{x}{2} \in \left(0, \frac{\pi}{2}\right) \subset \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)\right)$$

$$\boxed{\tan^{-1}\left(\frac{\sqrt{1-\cos x}}{\sqrt{1+\cos x}}\right) = \frac{x}{2}}$$

Question 2: Write in the simplest form: $\tan^{-1}\left(\frac{\cos x}{1+\sin x}\right)$, $-\frac{\pi}{2} < x < \frac{\pi}{2}$.

Solution: Write $\cos x = \sin\left(\frac{\pi}{2} - x\right)$ and $1+\sin x = 1+\cos\left(\frac{\pi}{2} - x\right)$. Using half-angle identities with $t = \frac{\pi}{2} - x$... more directly, use $\cos x = \cos^2\frac{x}{2} - \sin^2\frac{x}{2}$ and $1+\sin x = \left(\cos\frac{x}{2} + \sin\frac{x}{2}\right)^2$:

$$\frac{\cos x}{1+\sin x} = \frac{\left(\cos\frac{x}{2} - \sin\frac{x}{2}\right)\left(\cos\frac{x}{2} + \sin\frac{x}{2}\right)}{\left(\cos\frac{x}{2} + \sin\frac{x}{2}\right)^2} = \frac{\cos\frac{x}{2} - \sin\frac{x}{2}}{\cos\frac{x}{2} + \sin\frac{x}{2}}$$

Dividing numerator and denominator by $\cos\frac{x}{2}$:

$$= \frac{1-\tan\frac{x}{2}}{1+\tan\frac{x}{2}} = \tan\left(\frac{\pi}{4} - \frac{x}{2}\right)$$

Therefore,

$$\tan^{-1}\left(\frac{\cos x}{1+\sin x}\right) = \tan^{-1}\left[\tan\left(\frac{\pi}{4} - \frac{x}{2}\right)\right] = \frac{\pi}{4} - \frac{x}{2}$$

(valid since $-\frac{\pi}{2} < x < \frac{\pi}{2} \Rightarrow \frac{\pi}{4} - \frac{x}{2} \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$)

$$\boxed{\tan^{-1}\left(\frac{\cos x}{1+\sin x}\right) = \frac{\pi}{4} - \frac{x}{2}}$$

Question 3: Write in the simplest form: $\tan^{-1}\left(\dfrac{x}{\sqrt{a^2-x^2}}\right)$, $|x|<a$.

Solution: Put $x = a\sin\theta$, so $\theta = \sin^{-1}\left(\frac{x}{a}\right)$.

$$\frac{x}{\sqrt{a^2-x^2}} = \frac{a\sin\theta}{\sqrt{a^2-a^2\sin^2\theta}} = \frac{a\sin\theta}{a\cos\theta} = \tan\theta$$

Therefore,

$$\tan^{-1}\left(\frac{x}{\sqrt{a^2-x^2}}\right) = \tan^{-1}(\tan\theta) = \theta$$

$$\boxed{\tan^{-1}\left(\frac{x}{\sqrt{a^2-x^2}}\right) = \sin^{-1}\left(\frac{x}{a}\right)}$$

Question 4: Write in the simplest form: $\tan^{-1}\left(\dfrac{3a^2x-x^3}{a^3-3ax^2}\right)$, $a>0$, $-\dfrac{a}{\sqrt{3}} \le x \le \dfrac{a}{\sqrt{3}}$.

Solution: Put $x = a\tan\theta$, so $\theta = \tan^{-1}\left(\frac{x}{a}\right)$.

$$\frac{3a^2x-x^3}{a^3-3ax^2} = \frac{3a^3\tan\theta - a^3\tan^3\theta}{a^3-3a^3\tan^2\theta} = \frac{3\tan\theta-\tan^3\theta}{1-3\tan^2\theta} = \tan(3\theta)$$

Therefore,

$$\tan^{-1}\left(\frac{3a^2x-x^3}{a^3-3ax^2}\right) = \tan^{-1}[\tan(3\theta)] = 3\theta$$

$$\boxed{\tan^{-1}\left(\frac{3a^2x-x^3}{a^3-3ax^2}\right) = 3\tan^{-1}\left(\frac{x}{a}\right)}$$

Question 5: Prove that $\tan^{-1}\left(\dfrac{1}{2}\right) + \tan^{-1}\left(\dfrac{1}{3}\right) = \dfrac{\pi}{4}$.

Solution: Using $\tan^{-1}x + \tan^{-1}y = \tan^{-1}\left(\dfrac{x+y}{1-xy}\right)$ (valid since $xy = \dfrac{1}{6} < 1$):

$$\tan^{-1}\left(\frac{1}{2}\right) + \tan^{-1}\left(\frac{1}{3}\right) = \tan^{-1}\left(\frac{\frac{1}{2}+\frac{1}{3}}{1-\frac{1}{2}\cdot\frac{1}{3}}\right) = \tan^{-1}\left(\frac{\frac{5}{6}}{\frac{5}{6}}\right) = \tan^{-1}(1) = \frac{\pi}{4}$$

Hence proved. ■

Question 6: Prove that $2\tan^{-1}\left(\frac{1}{3}\right) + \tan^{-1}\left(\frac{1}{7}\right) = \frac{\pi}{4}$.

Solution: First simplify $2\tan^{-1}\left(\frac{1}{3}\right)$ using $2\tan^{-1} x = \tan^{-1}\left(\frac{2x}{1-x^2}\right)$:

$$2\tan^{-1}\left(\frac{1}{3}\right) = \tan^{-1}\left(\frac{2\cdot\frac{1}{3}}{1-\frac{1}{9}}\right) = \tan^{-1}\left(\frac{\frac{2}{3}}{\frac{8}{9}}\right) = \tan^{-1}\left(\frac{3}{4}\right)$$

Now add $\tan^{-1}\left(\frac{1}{7}\right)$:

$$\tan^{-1}\left(\frac{3}{4}\right) + \tan^{-1}\left(\frac{1}{7}\right) = \tan^{-1}\left(\frac{\frac{3}{4}+\frac{1}{7}}{1-\frac{3}{4}\cdot\frac{1}{7}}\right) = \tan^{-1}\left(\frac{\frac{25}{28}}{\frac{25}{28}}\right) = \tan^{-1}(1) = \frac{\pi}{4}$$

Hence proved. ■

Question 7: Prove that $\tan^{-1}\left(\frac{1}{5}\right) + \tan^{-1}\left(\frac{1}{7}\right) + \tan^{-1}\left(\frac{1}{3}\right) + \tan^{-1}\left(\frac{1}{8}\right) = \frac{\pi}{4}$.

Solution: Group the terms in pairs.

$$\tan^{-1}\left(\frac{1}{5}\right) + \tan^{-1}\left(\frac{1}{7}\right) = \tan^{-1}\left(\frac{\frac{1}{5}+\frac{1}{7}}{1-\frac{1}{35}}\right) = \tan^{-1}\left(\frac{\frac{12}{35}}{\frac{34}{35}}\right) = \tan^{-1}\left(\frac{6}{17}\right)$$

$$\tan^{-1}\left(\frac{1}{3}\right) + \tan^{-1}\left(\frac{1}{8}\right) = \tan^{-1}\left(\frac{\frac{1}{3}+\frac{1}{8}}{1-\frac{1}{24}}\right) = \tan^{-1}\left(\frac{\frac{11}{24}}{\frac{23}{24}}\right) = \tan^{-1}\left(\frac{11}{23}\right)$$

Now add the two results:

$$\tan^{-1}\left(\frac{6}{17}\right) + \tan^{-1}\left(\frac{11}{23}\right) = \tan^{-1}\left(\frac{\frac{6}{17}+\frac{11}{23}}{1-\frac{6}{17}\cdot\frac{11}{23}}\right) = \tan^{-1}\left(\frac{\frac{325}{391}}{\frac{325}{391}}\right) = \tan^{-1}(1) = \frac{\pi}{4}$$

Hence proved. ■

Question 8: Prove that $\cos^{-1}\left(\frac{4}{5}\right) + \cos^{-1}\left(\frac{12}{13}\right) = \cos^{-1}\left(\frac{33}{65}\right)$.

Solution: Let $A = \cos^{-1}\left(\frac{4}{5}\right)$ and $B = \cos^{-1}\left(\frac{12}{13}\right)$, so $\cos A = \frac{4}{5}$, $\cos B = \frac{12}{13}$.

Since $A, B \in [0, \pi]$ (and cosine is positive here), $\sin A = \frac{3}{5}$, $\sin B = \frac{5}{13}$.
Using $\cos(A+B) = \cos A \cos B - \sin A \sin B$:

$$\cos(A+B) = \frac{4}{5} \cdot \frac{12}{13} - \frac{3}{5} \cdot \frac{5}{13} = \frac{48}{65} - \frac{15}{65} = \frac{33}{65}$$

Since $A + B \in [0, \pi]$, taking $\cos^{-1}$ of both sides:

$$A + B = \cos^{-1}\left(\frac{33}{65}\right)$$

Hence proved. ■

Question 9: Prove that $\cos^{-1}\left(\frac{12}{13}\right) + \sin^{-1}\left(\frac{3}{5}\right) = \sin^{-1}\left(\frac{56}{65}\right)$.

Solution: Let $A = \cos^{-1}\left(\frac{12}{13}\right)$ and $B = \sin^{-1}\left(\frac{3}{5}\right)$, so $\cos A = \frac{12}{13}$, $\sin A = \frac{5}{13}$ and $\sin B = \frac{3}{5}$, $\cos B = \frac{4}{5}$.
Using $\sin(A+B) = \sin A \cos B + \cos A \sin B$:

$$\sin(A+B) = \frac{5}{13} \cdot \frac{4}{5} + \frac{12}{13} \cdot \frac{3}{5} = \frac{20}{65} + \frac{36}{65} = \frac{56}{65}$$

Hence,

$$A + B = \sin^{-1}\left(\frac{56}{65}\right)$$

Hence proved. ■

Question 10: Prove that $\sin^{-1}\left(\frac{8}{17}\right) + \sin^{-1}\left(\frac{3}{5}\right) = \tan^{-1}\left(\frac{77}{36}\right)$.

Solution: Let $A = \sin^{-1}\left(\frac{8}{17}\right)$ and $B = \sin^{-1}\left(\frac{3}{5}\right)$.
Then $\sin A = \frac{8}{17} \Rightarrow \cos A = \frac{15}{17} \Rightarrow \tan A = \frac{8}{15}$, and $\sin B = \frac{3}{5} \Rightarrow \cos B = \frac{4}{5} \Rightarrow \tan B = \frac{3}{4}$.
Using $\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$:

$$\tan(A+B) = \frac{\frac{8}{15} + \frac{3}{4}}{1 - \frac{8}{15} \cdot \frac{3}{4}} = \frac{\frac{32+45}{60}}{\frac{60-24}{60}} = \frac{\frac{77}{60}}{\frac{36}{60}} = \frac{77}{36}$$

Since A, B are both acute (positive sine, positive cosine), $A + B \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, so:

$$A + B = \tan^{-1}\left(\frac{77}{36}\right)$$

Hence proved. ■

Question 11: Prove that $\tan^{-1}\left(\frac{63}{16}\right) = \sin^{-1}\left(\frac{5}{13}\right) + \cos^{-1}\left(\frac{3}{5}\right)$.

Solution: Let $A = \sin^{-1}\left(\frac{5}{13}\right)$ and $B = \cos^{-1}\left(\frac{3}{5}\right)$.

Then $\sin A = \frac{5}{13} \Rightarrow \cos A = \frac{12}{13} \Rightarrow \tan A = \frac{5}{12}$, and $\cos B = \frac{3}{5} \Rightarrow \sin B = \frac{4}{5} \Rightarrow \tan B = \frac{4}{3}$.

$$\tan(A+B) = \frac{\frac{5}{12} + \frac{4}{3}}{1 - \frac{5}{12} \cdot \frac{4}{3}} = \frac{\frac{5+16}{12}}{\frac{36-20}{36}} = \frac{\frac{21}{12}}{\frac{16}{36}} = \frac{21}{12} \times \frac{36}{16} = \frac{63}{16}$$

So $A + B = \tan^{-1}\left(\frac{63}{16}\right)$, i.e.

$$\tan^{-1}\left(\frac{63}{16}\right) = \sin^{-1}\left(\frac{5}{13}\right) + \cos^{-1}\left(\frac{3}{5}\right)$$

Hence proved. ■

Question 12: Prove that $2\sin^{-1}\left(\frac{3}{5}\right) = \tan^{-1}\left(\frac{24}{7}\right)$.

Solution: Let $\theta = \sin^{-1}\left(\frac{3}{5}\right)$, so $\sin\theta = \frac{3}{5}$, and hence $\cos\theta = \frac{4}{5}$, $\tan\theta = \frac{3}{4}$.

Using $\tan(2\theta) = \frac{2\tan\theta}{1-\tan^2\theta}$:

$$\tan(2\theta) = \frac{2 \cdot \frac{3}{4}}{1 - \frac{9}{16}} = \frac{\frac{3}{2}}{\frac{7}{16}} = \frac{3}{2} \times \frac{16}{7} = \frac{24}{7}$$

Since $\theta = \sin^{-1}\left(\frac{3}{5}\right) \in \left(0, \frac{\pi}{2}\right)$, $2\theta \in (0, \pi)$; and because $\tan(2\theta) > 0$, in fact $2\theta \in \left(0, \frac{\pi}{2}\right)$. Hence:

$$2\theta = \tan^{-1}\left(\frac{24}{7}\right) \quad \Longrightarrow \quad 2\sin^{-1}\left(\frac{3}{5}\right) = \tan^{-1}\left(\frac{24}{7}\right)$$

Hence proved. ■

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3 Miscellaneous Exercise on Chapter 2

Question 1: Find the value of $\cos^{-1}\left(\cos\dfrac{13\pi}{6}\right)$.

Solution: We know $\cos^{-1}(\cos x) = x$ only when $x \in [0, \pi]$. Here $\dfrac{13\pi}{6} \notin [0, \pi]$, so rewrite it first.

$$\frac{13\pi}{6} = 2\pi + \frac{\pi}{6} \;\Rightarrow\; \cos\frac{13\pi}{6} = \cos\left(2\pi + \frac{\pi}{6}\right) = \cos\frac{\pi}{6}$$

Since $\dfrac{\pi}{6} \in [0, \pi]$,

$$\cos^{-1}\left(\cos\frac{13\pi}{6}\right) = \cos^{-1}\left(\cos\frac{\pi}{6}\right) = \frac{\pi}{6}$$

Question 2: Find the value of $\tan^{-1}\left(\tan\dfrac{7\pi}{6}\right)$.

Solution: We know $\tan^{-1}(\tan x) = x$ only when $x \in \left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$. Here $\dfrac{7\pi}{6}$ is outside this interval, so rewrite it.

$$\frac{7\pi}{6} = \pi + \frac{\pi}{6} \;\Rightarrow\; \tan\frac{7\pi}{6} = \tan\left(\pi + \frac{\pi}{6}\right) = \tan\frac{\pi}{6}$$

Since $\dfrac{\pi}{6} \in \left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$,

$$\tan^{-1}\left(\tan\frac{7\pi}{6}\right) = \tan^{-1}\left(\tan\frac{\pi}{6}\right) = \frac{\pi}{6}$$

Question 3: Prove that $2\sin^{-1}\left(\dfrac{3}{5}\right) - \tan^{-1}\left(\dfrac{17}{31}\right) = \dfrac{\pi}{4}$.

Solution: From Exercise 2.2, Q12, we already showed $2\sin^{-1}\left(\dfrac{3}{5}\right) = \tan^{-1}\left(\dfrac{24}{7}\right)$.

So we must show $\tan^{-1}\left(\dfrac{24}{7}\right) - \tan^{-1}\left(\dfrac{17}{31}\right) = \dfrac{\pi}{4}$.

Using $\tan^{-1} x - \tan^{-1} y = \tan^{-1}\left(\dfrac{x-y}{1+xy}\right)$:

$$\tan^{-1}\left(\frac{24}{7}\right) - \tan^{-1}\left(\frac{17}{31}\right) = \tan^{-1}\left(\frac{\frac{24}{7} - \frac{17}{31}}{1 + \frac{24}{7}\cdot\frac{17}{31}}\right) = \tan^{-1}\left(\frac{\frac{744-119}{217}}{\frac{217+408}{217}}\right) = \tan^{-1}\left(\frac{625}{625}\right) = \tan^{-1}(1) = \frac{\pi}{4}$$

Hence proved. ■

Question 4: Prove that $\sin^{-1}\left(\frac{4}{5}\right) + 2\tan^{-1}\left(\frac{1}{3}\right) = \frac{\pi}{2}$.

Solution: Let $\theta = \sin^{-1}\left(\frac{4}{5}\right)$, so $\sin\theta = \frac{4}{5}$, $\cos\theta = \frac{3}{5}$, $\tan\theta = \frac{4}{3}$, i.e. $\theta = \tan^{-1}\left(\frac{4}{3}\right)$.

From Q6 (Exercise 2.2 method), $2\tan^{-1}\left(\frac{1}{3}\right) = \tan^{-1}\left(\frac{3}{4}\right)$ (using $2\tan^{-1}x = \tan^{-1}\frac{2x}{1-x^2}$: $\frac{2/3}{1-1/9} = \frac{2/3}{8/9} = \frac{3}{4}$).

So we need: $\tan^{-1}\left(\frac{4}{3}\right) + \tan^{-1}\left(\frac{3}{4}\right)$.

Since $\frac{4}{3} \times \frac{3}{4} = 1$, and both are positive, using $\tan^{-1}x + \tan^{-1}\left(\frac{1}{x}\right) = \frac{\pi}{2}$ for $x > 0$:

$$\tan^{-1}\left(\frac{4}{3}\right) + \tan^{-1}\left(\frac{3}{4}\right) = \frac{\pi}{2}$$

Hence, $\sin^{-1}\left(\frac{4}{5}\right) + 2\tan^{-1}\left(\frac{1}{3}\right) = \frac{\pi}{2}$. Hence proved. ■

Question 5: Prove that $\tan^{-1}\sqrt{x} = \frac{1}{2}\cos^{-1}\left(\frac{1-x}{1+x}\right)$, $x \in [0, 1]$.

Solution: Put $x = \tan^2\theta$, so $\theta = \tan^{-1}\sqrt{x}$, where $\theta \in \left[0, \frac{\pi}{4}\right]$ (since $x \geq 0$).

$$\text{RHS argument:} \quad \frac{1-x}{1+x} = \frac{1-\tan^2\theta}{1+\tan^2\theta} = \cos 2\theta$$

So,

$$\frac{1}{2}\cos^{-1}(\cos 2\theta) = \frac{1}{2}(2\theta) = \theta \qquad (\text{valid since } 2\theta \in [0, \pi])$$

Therefore,

$$\frac{1}{2}\cos^{-1}\left(\frac{1-x}{1+x}\right) = \theta = \tan^{-1}\sqrt{x}$$

Hence proved. ■

Question 6: Prove that $\cot^{-1}\left(\frac{\sqrt{1+\sin x} + \sqrt{1-\sin x}}{\sqrt{1+\sin x} - \sqrt{1-\sin x}}\right) = \frac{x}{2}$, $x \in \left(0, \frac{\pi}{4}\right)$.

Solution: Since $0 < x < \frac{\pi}{4}$, we have $0 < \frac{x}{2} < \frac{\pi}{8}$, so $\cos\frac{x}{2} > \sin\frac{x}{2} > 0$.

Write $1 + \sin x = \left(\cos\frac{x}{2} + \sin\frac{x}{2}\right)^2$ and $1 - \sin x = \left(\cos\frac{x}{2} - \sin\frac{x}{2}\right)^2$, so

$$\sqrt{1+\sin x} = \cos\frac{x}{2} + \sin\frac{x}{2}, \qquad \sqrt{1-\sin x} = \cos\frac{x}{2} - \sin\frac{x}{2}$$

Substituting,

$$\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}} = \frac{2\cos\frac{x}{2}}{2\sin\frac{x}{2}} = \cot\frac{x}{2}$$

Therefore,

$$\cot^{-1}\left(\cot\frac{x}{2}\right) = \frac{x}{2} \qquad \left(\text{since } \frac{x}{2} \in (0,\pi)\right)$$

Hence proved. ■

Question 7: Prove that $\tan^{-1}\left(\dfrac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}\right) = \dfrac{\pi}{4} - \dfrac{1}{2}\cos^{-1}x,\ -\dfrac{1}{\sqrt{2}} \le x \le 1.$

Solution: Put $x = \cos 2\theta$, so $\theta = \dfrac{1}{2}\cos^{-1}x$, with $\theta \in \left[0, \dfrac{\pi}{4}\right]$ (matching the given range of x).
Then $1+x = 1+\cos 2\theta = 2\cos^2\theta$ and $1-x = 1-\cos 2\theta = 2\sin^2\theta$, so

$$\sqrt{1+x} = \sqrt{2}\cos\theta, \qquad \sqrt{1-x} = \sqrt{2}\sin\theta$$

Substituting:

$$\frac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}} = \frac{\sqrt{2}\cos\theta-\sqrt{2}\sin\theta}{\sqrt{2}\cos\theta+\sqrt{2}\sin\theta} = \frac{\cos\theta-\sin\theta}{\cos\theta+\sin\theta} = \frac{1-\tan\theta}{1+\tan\theta} = \tan\left(\frac{\pi}{4}-\theta\right)$$

Therefore,

$$\tan^{-1}\left(\frac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}\right) = \frac{\pi}{4}-\theta = \frac{\pi}{4}-\frac{1}{2}\cos^{-1}x$$

Hence proved. ■

Question 8: Prove that $\cos^{-1}(x) + \cos^{-1}\left(\dfrac{x}{2}+\dfrac{\sqrt{3-3x^2}}{2}\right) = \dfrac{\pi}{3},\ x \in \left[\dfrac{1}{2}, 1\right].$

Solution: Let $A = \cos^{-1}x$, so $\cos A = x$ and, since $A \in [0,\pi]$ and $x \ge \dfrac{1}{2}$, we have $A \in \left[0, \dfrac{\pi}{3}\right]$, so $\sin A = \sqrt{1-x^2} \ge 0$.
Consider $\cos\left(\dfrac{\pi}{3}-A\right)$:

$$\cos\left(\frac{\pi}{3}-A\right) = \cos\frac{\pi}{3}\cos A + \sin\frac{\pi}{3}\sin A = \frac{1}{2}\cdot x + \frac{\sqrt{3}}{2}\cdot\sqrt{1-x^2} = \frac{x}{2}+\frac{\sqrt{3(1-x^2)}}{2} = \frac{x}{2}+\frac{\sqrt{3-3x^2}}{2}$$

Since $\dfrac{\pi}{3}-A \in [0,\pi]$ for the given range of x, taking $\cos^{-1}$ of both sides:

$$\cos^{-1}\left(\frac{x}{2}+\frac{\sqrt{3-3x^2}}{2}\right) = \frac{\pi}{3}-A = \frac{\pi}{3}-\cos^{-1}x$$

Rearranging,

$$\cos^{-1} x + \cos^{-1}\left(\frac{x}{2} + \frac{\sqrt{3-3x^2}}{2}\right) = \frac{\pi}{3}$$

Hence proved. ■

Question 9: Solve for x: $\tan^{-1}(2x) + \tan^{-1}(3x) = \frac{\pi}{4}$.

Solution: Using $\tan^{-1} a + \tan^{-1} b = \tan^{-1}\left(\frac{a+b}{1-ab}\right)$:

$$\tan^{-1}\left(\frac{2x+3x}{1-6x^2}\right) = \frac{\pi}{4} \Rightarrow \frac{5x}{1-6x^2} = 1$$

$$5x = 1 - 6x^2 \Rightarrow 6x^2 + 5x - 1 = 0 \Rightarrow (6x-1)(x+1) = 0 \Rightarrow x = \frac{1}{6} \text{ or } x = -1$$

Checking: If $x = -1$: $\tan^{-1}(-2) + \tan^{-1}(-3)$ is negative (sum of two negative angles), which cannot equal $\frac{\pi}{4}$ (positive). So $x = -1$ is rejected.

If $x = \frac{1}{6}$: $2x = \frac{1}{3}$, $3x = \frac{1}{2}$, both positive and $2x \cdot 3x = \frac{1}{6} < 1$, so the formula applies validly and gives $\frac{\pi}{4}$. Valid.

$$\boxed{x = \frac{1}{6}}$$

Question 10: Solve: $2\tan^{-1}(\cos x) = \tan^{-1}(2\operatorname{cosec} x)$.

Solution: Using $2\tan^{-1} y = \tan^{-1}\left(\frac{2y}{1-y^2}\right)$ with $y = \cos x$:

$$2\tan^{-1}(\cos x) = \tan^{-1}\left(\frac{2\cos x}{1-\cos^2 x}\right) = \tan^{-1}\left(\frac{2\cos x}{\sin^2 x}\right)$$

Setting this equal to $\tan^{-1}(2\operatorname{cosec} x) = \tan^{-1}\left(\frac{2}{\sin x}\right)$:

$$\frac{2\cos x}{\sin^2 x} = \frac{2}{\sin x} \Rightarrow \frac{\cos x}{\sin x} = 1 \Rightarrow \cot x = 1$$

$$\boxed{x = \frac{\pi}{4}}$$

Question 11: Solve for x ($x > 0$): $\tan^{-1}\left(\dfrac{1-x}{1+x}\right) = \dfrac{1}{2}\tan^{-1}x$.

Solution: For $x > 0$, recall the identity $\tan^{-1}\left(\dfrac{1-x}{1+x}\right) = \dfrac{\pi}{4} - \tan^{-1}x$ (a special case of the subtraction formula with $\tan^{-1}1 = \dfrac{\pi}{4}$).
So the equation becomes:

$$\frac{\pi}{4} - \tan^{-1}x = \frac{1}{2}\tan^{-1}x \;\Rightarrow\; \frac{\pi}{4} = \frac{3}{2}\tan^{-1}x \;\Rightarrow\; \tan^{-1}x = \frac{\pi}{6}$$

$$\boxed{x = \tan\frac{\pi}{6} = \frac{1}{\sqrt{3}}}$$

Question 12: $\tan^{-1}\sqrt{3} - \cot^{-1}(-\sqrt{3})$ is equal to:
(A) π (B) $-\dfrac{\pi}{2}$ (C) 0 (D) $2\sqrt{3}$

Solution:

$$\tan^{-1}\sqrt{3} = \frac{\pi}{3}$$

For $\cot^{-1}(-\sqrt{3})$, use $\cot^{-1}(-x) = \pi - \cot^{-1}x$:

$$\cot^{-1}(-\sqrt{3}) = \pi - \cot^{-1}(\sqrt{3}) = \pi - \frac{\pi}{6} = \frac{5\pi}{6}$$

Therefore,

$$\tan^{-1}\sqrt{3} - \cot^{-1}(-\sqrt{3}) = \frac{\pi}{3} - \frac{5\pi}{6} = \frac{2\pi - 5\pi}{6} = -\frac{3\pi}{6} = -\frac{\pi}{2}$$

$$\boxed{\text{Option (B) is correct}}$$

Question 13: If $\sin^{-1}(1-x) - 2\sin^{-1}x = \dfrac{\pi}{2}$, then x equals:
(A) $0, \dfrac{1}{2}$ (B) $1, \dfrac{1}{2}$ (C) 0 (D) $\dfrac{1}{2}$

Solution: Rewrite: $\sin^{-1}(1-x) = \dfrac{\pi}{2} + 2\sin^{-1}x$. Taking sin of both sides:

$$1 - x = \sin\left(\frac{\pi}{2} + 2\sin^{-1}x\right) = \cos(2\sin^{-1}x) = 1 - 2\sin^2(\sin^{-1}x) = 1 - 2x^2$$

$$1 - x = 1 - 2x^2 \;\Rightarrow\; 2x^2 - x = 0 \;\Rightarrow\; x(2x-1) = 0 \;\Rightarrow\; x = 0 \text{ or } x = \frac{1}{2}$$

Verification (essential, since taking sine can introduce extraneous roots):

For $x = 0$: LHS $= \sin^{-1}(1) - 2\sin^{-1}(0) = \frac{\pi}{2} - 0 = \frac{\pi}{2}$. ✓ Valid.

For $x = \frac{1}{2}$: LHS $= \sin^{-1}\left(\frac{1}{2}\right) - 2\sin^{-1}\left(\frac{1}{2}\right) = -\sin^{-1}\left(\frac{1}{2}\right) = -\frac{\pi}{6} \neq \frac{\pi}{2}$. Rejected (extraneous).

So the only valid solution is $x = 0$.

$$\boxed{\text{Option (C) is correct}}$$

Question 14: $\sec^2\big(\tan^{-1} 2\big) + \operatorname{cosec}^2\big(\cot^{-1} 3\big)$ is equal to:
(A) 5 (B) 13 (C) 15 (D) 19

Solution: Let $\theta = \tan^{-1} 2$, so $\tan\theta = 2$. Using $\sec^2\theta = 1 + \tan^2\theta$:

$$\sec^2(\tan^{-1} 2) = 1 + 2^2 = 5$$

Let $\phi = \cot^{-1} 3$, so $\cot\phi = 3$. Using $\operatorname{cosec}^2\phi = 1 + \cot^2\phi$:

$$\operatorname{cosec}^2(\cot^{-1} 3) = 1 + 3^2 = 10$$

Adding,

$$\sec^2(\tan^{-1} 2) + \operatorname{cosec}^2(\cot^{-1} 3) = 5 + 10 = 15$$

$$\boxed{\text{Option (C) is correct}}$$

Quick Revision: Formula Sheet

1. Principal value ranges

$$\sin^{-1} x \in \left[-\tfrac{\pi}{2}, \tfrac{\pi}{2}\right], \quad \cos^{-1} x \in [0, \pi], \quad \tan^{-1} x \in \left(-\tfrac{\pi}{2}, \tfrac{\pi}{2}\right)$$

$$\cot^{-1} x \in (0, \pi), \quad \sec^{-1} x \in [0, \pi] - \{\tfrac{\pi}{2}\}, \quad \mathrm{cosec}^{-1}x \in \left[-\tfrac{\pi}{2}, \tfrac{\pi}{2}\right] - \{0\}$$

2. Reciprocal identities

$$\sin^{-1} \frac{1}{x} = \mathrm{cosec}^{-1}x, \quad \cos^{-1} \frac{1}{x} = \sec^{-1} x, \quad \tan^{-1} \frac{1}{x} = \cot^{-1} x \ (x > 0)$$

3. Negative-argument identities

$$\sin^{-1}(-x) = -\sin^{-1} x, \quad \cos^{-1}(-x) = \pi - \cos^{-1} x, \quad \tan^{-1}(-x) = -\tan^{-1} x$$

$$\cot^{-1}(-x) = \pi - \cot^{-1} x, \quad \sec^{-1}(-x) = \pi - \sec^{-1} x, \quad \mathrm{cosec}^{-1}(-x) = -\mathrm{cosec}^{-1}x$$

4. Complementary-angle identities

$$\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}, \quad \tan^{-1} x + \cot^{-1} x = \frac{\pi}{2}, \quad \sec^{-1} x + \mathrm{cosec}^{-1}x = \frac{\pi}{2}$$

5. Sum and difference formulas

$$\tan^{-1} x + \tan^{-1} y = \tan^{-1}\left(\frac{x+y}{1-xy}\right) \ (xy < 1), \qquad \tan^{-1} x - \tan^{-1} y = \tan^{-1}\left(\frac{x-y}{1+xy}\right) \ (xy > -1)$$

6. Double-angle formulas

$$2\tan^{-1} x = \sin^{-1}\left(\frac{2x}{1+x^2}\right) = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right) = \tan^{-1}\left(\frac{2x}{1-x^2}\right)$$

$$2\sin^{-1} x = \sin^{-1}\left(2x\sqrt{1-x^2}\right), \quad 2\cos^{-1} x = \cos^{-1}(2x^2 - 1)$$

7. Triple-angle formulas

$$3\sin^{-1} x = \sin^{-1}(3x - 4x^3), \qquad 3\cos^{-1} x = \cos^{-1}(4x^3 - 3x)$$

$$3\tan^{-1} x = \tan^{-1}\left(\frac{3x - x^3}{1 - 3x^2}\right)$$

Common mistakes to avoid

- Never write $\sin^{-1}(\sin x) = x$ or $\cos^{-1}(\cos x) = x$ blindly — this only holds when x lies within the function's *principal value branch*. Outside it, first reduce x using periodicity/symmetry identities.
- $\tan^{-1} x + \tan^{-1} y = \tan^{-1}\left(\frac{x+y}{1-xy}\right)$ needs an adjustment of $\pm\pi$ when $xy > 1$ — always check the quadrant of the final angle.
- $\sin^{-1} x$ is **not** the same as $\dfrac{1}{\sin x}$ — the -1 is a function label, not an exponent.

— End of Chapter 2: Inverse Trigonometric Functions —
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