NCERT Solutions

Class 12 Mathematics

Chapter 3: Matrices

Complete Step-by-Step Solutions

Exercises 3.1, 3.2, 3.3, 3.4 & Miscellaneous Exercise

- All 77 questions solved with detailed, step-by-step working
- Clear mathematical notation, easy-to-follow explanations
- Ideal for CBSE Board exam preparation
- Print-friendly, distraction-free layout

Quick Revision: Key Concepts

- **Matrix:** A rectangular arrangement of numbers (or functions) in rows and columns. A matrix with m rows and n columns is said to have order $m \times n$, and has mn elements.
- **Types of matrices:** Row matrix, Column matrix, Square matrix ($m = n$), Diagonal matrix, Scalar matrix, Identity matrix, Zero (Null) matrix.
- **Equality of matrices:** Two matrices are equal if they have the same order and all corresponding elements are equal.
- **Addition/Subtraction:** Defined only for matrices of the *same order*; add/subtract corresponding elements.
- **Scalar multiplication:** Multiply every element of the matrix by the scalar.
- **Matrix multiplication:** AB is defined only if number of columns of A = number of rows of B. If A is $m \times n$ and B is $n \times p$, then AB is $m \times p$, with $(AB)_{ij} = \sum_k a_{ik}b_{kj}$. In general, $AB \neq BA$.
- **Transpose:** A' or A^T is obtained by interchanging rows and columns. Properties: $(A')' = A,\ (A+B)' = A' + B',\ (kA)' = kA',\ (AB)' = B'A'$.
- **Symmetric matrix:** $A' = A$. **Skew-symmetric matrix:** $A' = -A$ (diagonal elements are always 0).
- Every square matrix can be expressed as the sum of a symmetric and a skew-symmetric matrix: $A = \frac{1}{2}(A + A') + \frac{1}{2}(A - A')$.
- **Elementary operations:** Row/column interchange, multiplying a row/column by a non-zero scalar, adding a scalar multiple of one row/column to another.
- **Invertible matrix:** A square matrix A is invertible if there exists B such that $AB = BA = I$; then $B = A^{-1}$. A matrix has at most one inverse. Inverse (if it exists) can be found using elementary row/column operations.

1 Exercise 3.1

Question 1

In the matrix $A = \begin{bmatrix} 2 & 5 & 19 & -7 \\ 35 & -2 & \frac{5}{2} & 12 \\ \sqrt{3} & 1 & -5 & 17 \end{bmatrix}$, write:
(i) The order of the matrix (ii) The number of elements
(iii) Write the elements $a_{13}, a_{21}, a_{33}, a_{24}, a_{23}$.

Solution: (i) The matrix has 3 rows and 4 columns, so its order is 3×4.
(ii) Number of elements $= 3 \times 4 = 12$.
(iii) $a_{13} = 19, \quad a_{21} = 35, \quad a_{33} = -5, \quad a_{24} = 12, \quad a_{23} = \dfrac{5}{2}$.

Question 2

If a matrix has 24 elements, what are the possible orders it can have? What if it has 13 elements?

Solution: The possible orders of a matrix with mn elements are all ordered pairs (m, n) such that mn equals that number, since m, n are positive integers.
24 elements: $24 = 1 \times 24 = 2 \times 12 = 3 \times 8 = 4 \times 6 = 6 \times 4 = 8 \times 3 = 12 \times 2 = 24 \times 1$. So possible orders are: $1 \times 24,\ 2 \times 12,\ 3 \times 8,\ 4 \times 6,\ 6 \times 4,\ 8 \times 3,\ 12 \times 2,\ 24 \times 1$ (8 orders in all).
13 elements: Since 13 is prime, $13 = 1 \times 13 = 13 \times 1$. So possible orders are: 1×13 and 13×1.

Question 3

If a matrix has 18 elements, what are the possible orders it can have? What if it has 5 elements?

Solution: **18 elements:** $18 = 1 \times 18 = 2 \times 9 = 3 \times 6 = 6 \times 3 = 9 \times 2 = 18 \times 1$. Possible orders: $1 \times 18,\ 2 \times 9,\ 3 \times 6,\ 6 \times 3,\ 9 \times 2,\ 18 \times 1$ (6 orders).
5 elements: Since 5 is prime, possible orders are 1×5 and 5×1.

Question 4

Construct a 2×2 matrix $A = [a_{ij}]$ whose elements are given by:
(i) $a_{ij} = \dfrac{(i+j)^2}{2}$ (ii) $a_{ij} = \dfrac{i}{j}$ (iii) $a_{ij} = \dfrac{(i+2j)^2}{2}$

Solution: **(i)** $a_{11} = \frac{(1+1)^2}{2} = 2,\ a_{12} = \frac{(1+2)^2}{2} = \frac{9}{2},\ a_{21} = \frac{(2+1)^2}{2} = \frac{9}{2},\ a_{22} = \frac{(2+2)^2}{2} = 8.$

$$A = \begin{bmatrix} 2 & \frac{9}{2} \\ \frac{9}{2} & 8 \end{bmatrix}$$

(ii) $a_{11} = \frac{1}{1} = 1,\ a_{12} = \frac{1}{2},\ a_{21} = \frac{2}{1} = 2,\ a_{22} = \frac{2}{2} = 1.$

$$A = \begin{bmatrix} 1 & \frac{1}{2} \\ 2 & 1 \end{bmatrix}$$

(iii) $a_{11} = \frac{(1+2)^2}{2} = \frac{9}{2},\ a_{12} = \frac{(1+4)^2}{2} = \frac{25}{2},\ a_{21} = \frac{(2+2)^2}{2} = 8,\ a_{22} = \frac{(2+4)^2}{2} = 18.$

$$A = \begin{bmatrix} \frac{9}{2} & \frac{25}{2} \\ 8 & 18 \end{bmatrix}$$

Question 5

Construct a 3×4 matrix whose elements are given by:
(i) $a_{ij} = \dfrac{1}{2}|-3i+j|$ (ii) $a_{ij} = 2i - j$

Solution: **(i)** Using $a_{ij} = \frac{1}{2}|-3i+j|$:

$a_{11} = 1,\ a_{12} = \frac{1}{2},\ a_{13} = 0,\ a_{14} = \frac{1}{2},\ a_{21} = \frac{5}{2},\ a_{22} = 2,\ a_{23} = \frac{3}{2},\ a_{24} = 1,\ a_{31} = 4,\ a_{32} = \frac{7}{2},\ a_{33} = 3,\ a_{34} = \frac{5}{2}$

$$A = \begin{bmatrix} 1 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{5}{2} & 2 & \frac{3}{2} & 1 \\ 4 & \frac{7}{2} & 3 & \frac{5}{2} \end{bmatrix}$$

(ii) Using $a_{ij} = 2i - j$:

$a_{11} = 1,\ a_{12} = 0,\ a_{13} = -1,\ a_{14} = -2,\ a_{21} = 3,\ a_{22} = 2,\ a_{23} = 1,\ a_{24} = 0,\ a_{31} = 5,\ a_{32} = 4,\ a_{33} = 3,\ a_{34} = 2$

$$A = \begin{bmatrix} 1 & 0 & -1 & -2 \\ 3 & 2 & 1 & 0 \\ 5 & 4 & 3 & 2 \end{bmatrix}$$

Question 6

Find the values of x, y, z from the following equations:
(i) $\begin{bmatrix} 4 & 3 \\ x & 5 \end{bmatrix} = \begin{bmatrix} y & z \\ 1 & 5 \end{bmatrix}$
(ii) $\begin{bmatrix} x+y & 2 \\ 5+z & xy \end{bmatrix} = \begin{bmatrix} 6 & 2 \\ 5 & 8 \end{bmatrix}$
(iii) $\begin{bmatrix} x+y+z \\ x+z \\ y+z \end{bmatrix} = \begin{bmatrix} 9 \\ 5 \\ 7 \end{bmatrix}$

Solution: **(i)** Comparing corresponding elements: $y = 4,\ z = 3,\ x = 1.$
(ii) Comparing: $x + y = 6,\ 5 + z = 5 \Rightarrow z = 0,\ xy = 8.$ From $x + y = 6$ and $xy = 8$: x and y are roots of $t^2 - 6t + 8 = 0 \Rightarrow (t-2)(t-4) = 0 \Rightarrow t = 2, 4.$ So $x = 2, y = 4$ or $x = 4, y = 2$, and $z = 0.$
(iii) $x + y + z = 9,\quad x + z = 5,\quad y + z = 7.$ Subtracting: $(x+y+z) - (x+z) = 9 - 5 \Rightarrow y = 4.$ $(x+y+z) - (y+z) = 9 - 7 \Rightarrow x = 2.$ From $x + z = 5 \Rightarrow z = 5 - 2 = 3.$ So $x = 2,\ y = 4,\ z = 3.$

Question 7

Find the value of a, b, c, d from the equation:

$$\begin{bmatrix} 2a+b & a-2b \\ 5c-d & 4c+3d \end{bmatrix} = \begin{bmatrix} 4 & -3 \\ 11 & 24 \end{bmatrix}$$

Solution: Comparing corresponding elements:

$$2a + b = 4 \quad (1) \qquad a - 2b = -3 \quad (2) \qquad 5c - d = 11 \quad (3) \qquad 4c + 3d = 24 \quad (4)$$

From (1): $b = 4-2a$. Substituting in (2): $a-2(4-2a) = -3 \Rightarrow a-8+4a = -3 \Rightarrow 5a = 5 \Rightarrow a = 1$. Then $b = 4 - 2(1) = 2$.
From (3): $d = 5c - 11$. Substituting in (4): $4c + 3(5c - 11) = 24 \Rightarrow 4c + 15c - 33 = 24 \Rightarrow 19c = 57 \Rightarrow c = 3$. Then $d = 5(3) - 11 = 4$.
Answer: $a = 1,\ b = 2,\ c = 3,\ d = 4$.

Question 8

$A = [a_{ij}]_{m \times n}$ is a square matrix if:
(A) $m < n$ (B) $m > n$ (C) $m = n$ (D) None of these

Solution: A square matrix has an equal number of rows and columns, i.e. $m = n$. **Correct answer: (C) $m = n$.**

Question 9

Which of the given values of x and y make the following pair of matrices equal?

$$\begin{bmatrix} 3x+7 & 5 \\ y+1 & 2-3x \end{bmatrix} = \begin{bmatrix} 0 & y-2 \\ 8 & 4 \end{bmatrix}$$

(A) $x = \dfrac{-1}{3},\ y = 7$ (B) Not possible to find
(C) $y = 7,\ x = \dfrac{-2}{3}$ (D) $x = \dfrac{-1}{3},\ y = \dfrac{-2}{3}$

Solution: Comparing corresponding elements:

$$3x + 7 = 0 \Rightarrow x = -\tfrac{7}{3} \qquad 2 - 3x = 4 \Rightarrow x = -\tfrac{2}{3}$$

Also $5 = y - 2 \Rightarrow y = 7$, and $y + 1 = 8 \Rightarrow y = 7$ (consistent).
Here $x = -\frac{7}{3}$ (from first entry) does not match $x = -\frac{2}{3}$ (from the $(2,2)$ entry); so no single value of x satisfies both equations simultaneously.
Correct answer: (B) It is not possible to find the values of x and y.

Question 10

The number of all possible matrices of order 3×3 with each entry 0 or 1 is:
(A) 27 (B) 18 (C) 81 (D) 512

Solution: A 3×3 matrix has 9 entries, each of which can independently be chosen in 2 ways (0 or 1). So the total number of such matrices $= 2^9 = 512$. **Correct answer: (D) 512.**

2 Exercise 3.2

Question 1

Let $A = \begin{bmatrix} 2 & 4 \\ 3 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 3 \\ -2 & 5 \end{bmatrix}$, $C = \begin{bmatrix} -2 & 5 \\ 3 & 4 \end{bmatrix}$. Find each of the following:
(i) $A + B$ (ii) $A - B$ (iii) $3A - C$ (iv) AB (v) BA

Solution: **(i)** $A + B = \begin{bmatrix} 2+1 & 4+3 \\ 3-2 & 2+5 \end{bmatrix} = \begin{bmatrix} 3 & 7 \\ 1 & 7 \end{bmatrix}$

(ii) $A - B = \begin{bmatrix} 2-1 & 4-3 \\ 3+2 & 2-5 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 5 & -3 \end{bmatrix}$

(iii) $3A = \begin{bmatrix} 6 & 12 \\ 9 & 6 \end{bmatrix}$, so $3A - C = \begin{bmatrix} 6-(-2) & 12-5 \\ 9-3 & 6-4 \end{bmatrix} = \begin{bmatrix} 8 & 7 \\ 6 & 2 \end{bmatrix}$

(iv) $AB = \begin{bmatrix} 2(1)+4(-2) & 2(3)+4(5) \\ 3(1)+2(-2) & 3(3)+2(5) \end{bmatrix} = \begin{bmatrix} 2-8 & 6+20 \\ 3-4 & 9+10 \end{bmatrix} = \begin{bmatrix} -6 & 26 \\ -1 & 19 \end{bmatrix}$

(v) $BA = \begin{bmatrix} 1(2)+3(3) & 1(4)+3(2) \\ -2(2)+5(3) & -2(4)+5(2) \end{bmatrix} = \begin{bmatrix} 2+9 & 4+6 \\ -4+15 & -8+10 \end{bmatrix} = \begin{bmatrix} 11 & 10 \\ 11 & 2 \end{bmatrix}$

Note that $AB \neq BA$, confirming matrix multiplication is not commutative in general.

Question 2

Compute the following:

(i) $\begin{bmatrix} a^2+b^2 & b^2+c^2 \\ a^2+c^2 & a^2+b^2 \end{bmatrix} + \begin{bmatrix} 2ab & 2bc \\ -2ac & -2ab \end{bmatrix}$

(ii) $\begin{bmatrix} a & b \\ -b & a \end{bmatrix} + \begin{bmatrix} a & b \\ b & a \end{bmatrix}$

(iii) $\begin{bmatrix} -1 & 4 & -6 \\ 8 & 5 & 16 \\ 2 & 8 & 5 \end{bmatrix} + \begin{bmatrix} 12 & 7 & 6 \\ 8 & 0 & 5 \\ 3 & 2 & 4 \end{bmatrix}$

(iv) $\begin{bmatrix} \cos^2 x & \sin^2 x \\ \sin^2 x & \cos^2 x \end{bmatrix} + \begin{bmatrix} \sin^2 x & \cos^2 x \\ \cos^2 x & \sin^2 x \end{bmatrix}$

Solution: **(i)** Adding corresponding elements:

$$\begin{bmatrix} a^2+b^2+2ab & b^2+c^2+2bc \\ a^2+c^2-2ac & a^2+b^2-2ab \end{bmatrix} = \begin{bmatrix} (a+b)^2 & (b+c)^2 \\ (a-c)^2 & (a-b)^2 \end{bmatrix}$$

(ii) $\begin{bmatrix} a+a & b+b \\ -b+b & a+a \end{bmatrix} = \begin{bmatrix} 2a & 2b \\ 0 & 2a \end{bmatrix}$

(iii) $\begin{bmatrix} -1+12 & 4+7 & -6+6 \\ 8+8 & 5+0 & 16+5 \\ 2+3 & 8+2 & 5+4 \end{bmatrix} = \begin{bmatrix} 11 & 11 & 0 \\ 16 & 5 & 21 \\ 5 & 10 & 9 \end{bmatrix}$

(iv) $\begin{bmatrix} \cos^2 x+\sin^2 x & \sin^2 x+\cos^2 x \\ \sin^2 x+\cos^2 x & \cos^2 x+\sin^2 x \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ (using $\sin^2 x + \cos^2 x = 1$)

Question 3

Compute the indicated products:

(i) $\begin{bmatrix} a & b \\ -b & a \end{bmatrix}\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ (ii) $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}\begin{bmatrix} 2 & 3 & 4 \end{bmatrix}$ (iii) $\begin{bmatrix} 1 & -2 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{bmatrix}$ (iv) $\begin{bmatrix} 2 & 3 & 4 \\ 3 & 4 & 5 \\ 4 & 5 & 6 \end{bmatrix}\begin{bmatrix} 1 & -3 & 5 \\ 0 & 2 & 4 \\ 3 & 0 & 5 \end{bmatrix}$ (v) $\begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix}$

Solution: **(i)** $\begin{bmatrix} a \cdot a + b \cdot b & a(-b) + b \cdot a \\ -b \cdot a + a \cdot b & -b(-b) + a \cdot a \end{bmatrix} = \begin{bmatrix} a^2 + b^2 & 0 \\ 0 & a^2 + b^2 \end{bmatrix}$

(ii) $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}\begin{bmatrix} 2 & 3 & 4 \end{bmatrix} = \begin{bmatrix} 1(2) & 1(3) & 1(4) \\ 2(2) & 2(3) & 2(4) \\ 3(2) & 3(3) & 3(4) \end{bmatrix} = \begin{bmatrix} 2 & 3 & 4 \\ 4 & 6 & 8 \\ 6 & 9 & 12 \end{bmatrix}$

(iii) $\begin{bmatrix} 1(1) + (-2)(2) & 1(2) + (-2)(3) & 1(3) + (-2)(1) \\ 2(1) + 3(2) & 2(2) + 3(3) & 2(3) + 3(1) \end{bmatrix} = \begin{bmatrix} 1-4 & 2-6 & 3-2 \\ 2+6 & 4+9 & 6+3 \end{bmatrix} = \begin{bmatrix} -3 & -4 & 1 \\ 8 & 13 & 9 \end{bmatrix}$

(iv) Row 1: $[2(1) + 3(0) + 4(3),\ 2(-3) + 3(2) + 4(0),\ 2(5) + 3(4) + 4(5)] = [14, 0, 42]$
Row 2: $[3(1) + 4(0) + 5(3),\ 3(-3) + 4(2) + 5(0),\ 3(5) + 4(4) + 5(5)] = [18, -1, 56]$
Row 3: $[4(1) + 5(0) + 6(3),\ 4(-3) + 5(2) + 6(0),\ 4(5) + 5(4) + 6(5)] = [22, -2, 70]$

$$= \begin{bmatrix} 14 & 0 & 42 \\ 18 & -1 & 56 \\ 22 & -2 & 70 \end{bmatrix}$$

(v) $\begin{bmatrix} 2(1) + 1(-1) & 2(-2) + 1(3) \\ 3(1) + 2(-1) & 3(-2) + 2(3) \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}$

Question 4

If $A = \begin{bmatrix} 1 & 2 & -3 \\ 5 & 0 & 2 \\ 1 & -1 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 3 & -1 & 2 \\ 4 & 2 & 5 \\ 2 & 0 & 3 \end{bmatrix}$, $C = \begin{bmatrix} 4 & 1 & 2 \\ 0 & 3 & 2 \\ 1 & -2 & 3 \end{bmatrix}$, find $A + B$ and $B - C$. Also, verify that $A + (B - C) = (A + B) - C$.

Solution:

$$A + B = \begin{bmatrix} 1+3 & 2-1 & -3+2 \\ 5+4 & 0+2 & 2+5 \\ 1+2 & -1+0 & 1+3 \end{bmatrix} = \begin{bmatrix} 4 & 1 & -1 \\ 9 & 2 & 7 \\ 3 & -1 & 4 \end{bmatrix}$$

$$B - C = \begin{bmatrix} 3-4 & -1-1 & 2-2 \\ 4-0 & 2-3 & 5-2 \\ 2-1 & 0+2 & 3-3 \end{bmatrix} = \begin{bmatrix} -1 & -2 & 0 \\ 4 & -1 & 3 \\ 1 & 2 & 0 \end{bmatrix}$$

Verification:

$$A + (B - C) = \begin{bmatrix} 1-1 & 2-2 & -3+0 \\ 5+4 & 0-1 & 2+3 \\ 1+1 & -1+2 & 1+0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -3 \\ 9 & -1 & 5 \\ 2 & 1 & 1 \end{bmatrix}$$

$$(A + B) - C = \begin{bmatrix} 4-4 & 1-1 & -1-2 \\ 9-0 & 2-3 & 7-2 \\ 3-1 & -1+2 & 4-3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -3 \\ 9 & -1 & 5 \\ 2 & 1 & 1 \end{bmatrix}$$

Since both results are equal, $A + (B - C) = (A + B) - C$ is verified.

Question 5

If $A = \begin{bmatrix} \frac{2}{3} & 1 & \frac{5}{3} \\ \frac{1}{3} & \frac{2}{3} & \frac{4}{3} \\ \frac{7}{3} & 2 & \frac{2}{3} \end{bmatrix}$ and $B = \begin{bmatrix} \frac{2}{5} & \frac{3}{5} & 1 \\ \frac{1}{5} & \frac{2}{5} & \frac{4}{5} \\ \frac{7}{5} & \frac{6}{5} & \frac{2}{5} \end{bmatrix}$, then compute $3A - 5B$.

Solution:

$$3A = \begin{bmatrix} 2 & 3 & 5 \\ 1 & 2 & 4 \\ 7 & 6 & 2 \end{bmatrix}, \qquad 5B = \begin{bmatrix} 2 & 3 & 5 \\ 1 & 2 & 4 \\ 7 & 6 & 2 \end{bmatrix}$$

$$3A - 5B = \begin{bmatrix} 2-2 & 3-3 & 5-5 \\ 1-1 & 2-2 & 4-4 \\ 7-7 & 6-6 & 2-2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = O$$

Question 6

Simplify: $\cos\theta \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} + \sin\theta \begin{bmatrix} \sin\theta & -\cos\theta \\ \cos\theta & \sin\theta \end{bmatrix}$

Solution:

$$= \begin{bmatrix} \cos^2\theta & \sin\theta\cos\theta \\ -\sin\theta\cos\theta & \cos^2\theta \end{bmatrix} + \begin{bmatrix} \sin^2\theta & -\sin\theta\cos\theta \\ \sin\theta\cos\theta & \sin^2\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2\theta + \sin^2\theta & \sin\theta\cos\theta - \sin\theta\cos\theta \\ -\sin\theta\cos\theta + \sin\theta\cos\theta & \cos^2\theta + \sin^2\theta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

Question 7

Find X and Y, if:

(i) $X + Y = \begin{bmatrix} 7 & 0 \\ 2 & 5 \end{bmatrix}$ and $X - Y = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$

(ii) $2X + 3Y = \begin{bmatrix} 2 & 3 \\ 4 & 0 \end{bmatrix}$ and $3X + 2Y = \begin{bmatrix} 2 & -2 \\ -1 & 5 \end{bmatrix}$

Solution: **(i)** Adding the two equations: $2X = \begin{bmatrix} 10 & 0 \\ 2 & 8 \end{bmatrix} \Rightarrow X = \begin{bmatrix} 5 & 0 \\ 1 & 4 \end{bmatrix}$. Subtracting: $2Y = \begin{bmatrix} 4 & 0 \\ 2 & 2 \end{bmatrix} \Rightarrow Y = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}$.

(ii) Multiply first equation by 3 and second by 2: $6X + 9Y = \begin{bmatrix} 6 & 9 \\ 12 & 0 \end{bmatrix}$, $\quad 6X + 4Y = \begin{bmatrix} 4 & -4 \\ -2 & 10 \end{bmatrix}$.

Subtracting: $5Y = \begin{bmatrix} 2 & 13 \\ 14 & -10 \end{bmatrix} \Rightarrow Y = \begin{bmatrix} \frac{2}{5} & \frac{13}{5} \\ \frac{14}{5} & -2 \end{bmatrix}$. Substituting back into $2X + 3Y = \begin{bmatrix} 2 & 3 \\ 4 & 0 \end{bmatrix}$:

$2X = \begin{bmatrix} 2 & 3 \\ 4 & 0 \end{bmatrix} - 3Y = \begin{bmatrix} 2 - \frac{6}{5} & 3 - \frac{39}{5} \\ 4 - \frac{42}{5} & 0 + 6 \end{bmatrix} = \begin{bmatrix} \frac{4}{5} & -\frac{24}{5} \\ -\frac{22}{5} & 6 \end{bmatrix}$

$$X = \begin{bmatrix} \frac{2}{5} & -\frac{12}{5} \\ -\frac{11}{5} & 3 \end{bmatrix}$$

Question 8

Find X, if $Y = \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix}$ and $2X + Y = \begin{bmatrix} 1 & 0 \\ -3 & 2 \end{bmatrix}$.

Solution: $2X = \begin{bmatrix} 1 & 0 \\ -3 & 2 \end{bmatrix} - \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} -2 & -2 \\ -4 & -2 \end{bmatrix}$

$$X = \begin{bmatrix} -1 & -1 \\ -2 & -1 \end{bmatrix}$$

Question 9

Find x and y, if $2\begin{bmatrix} 1 & 3 \\ 0 & x \end{bmatrix} + \begin{bmatrix} y & 0 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 6 \\ 1 & 8 \end{bmatrix}$.

Solution:

$$\begin{bmatrix} 2 & 6 \\ 0 & 2x \end{bmatrix} + \begin{bmatrix} y & 0 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 6 \\ 1 & 8 \end{bmatrix} \Rightarrow \begin{bmatrix} 2+y & 6 \\ 1 & 2x+2 \end{bmatrix} = \begin{bmatrix} 5 & 6 \\ 1 & 8 \end{bmatrix}$$

Comparing: $2 + y = 5 \Rightarrow y = 3$, and $2x + 2 = 8 \Rightarrow x = 3$.

Question 10

Solve the matrix equation for x, y, z, t if $2\begin{bmatrix} x & z \\ y & t \end{bmatrix} + 3\begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix} = 3\begin{bmatrix} 3 & 5 \\ 4 & 6 \end{bmatrix}$.

Solution:

$$2\begin{bmatrix} x & z \\ y & t \end{bmatrix} = \begin{bmatrix} 9 & 15 \\ 12 & 18 \end{bmatrix} - \begin{bmatrix} 3 & -3 \\ 0 & 6 \end{bmatrix} = \begin{bmatrix} 6 & 18 \\ 12 & 12 \end{bmatrix} \Rightarrow \begin{bmatrix} x & z \\ y & t \end{bmatrix} = \begin{bmatrix} 3 & 9 \\ 6 & 6 \end{bmatrix}$$

So $x = 3,\ z = 9,\ y = 6,\ t = 6$.

Question 11

If $x\begin{bmatrix} 2 \\ 3 \end{bmatrix} + y\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 10 \\ 5 \end{bmatrix}$, find the values of x and y.

Solution:

$$\begin{bmatrix} 2x - y \\ 3x + y \end{bmatrix} = \begin{bmatrix} 10 \\ 5 \end{bmatrix} \Rightarrow 2x - y = 10, \quad 3x + y = 5$$

Adding: $5x = 15 \Rightarrow x = 3$. Then $y = 2x - 10 = 6 - 10 = -4$. So $x = 3,\ y = -4$.

Question 12

Given $3\begin{bmatrix} x & y \\ z & w \end{bmatrix} = \begin{bmatrix} x & 6 \\ -1 & 2w \end{bmatrix} + \begin{bmatrix} 4 & x+y \\ z+w & 3 \end{bmatrix}$, find the values of x, y, z, w.

Solution: Right side $= \begin{bmatrix} x+4 & 6+x+y \\ -1+z+w & 2w+3 \end{bmatrix}$. Equating with $\begin{bmatrix} 3x & 3y \\ 3z & 3w \end{bmatrix}$:

$$3x = x + 4 \Rightarrow x = 2 \qquad 3y = 6 + x + y \Rightarrow 2y = 6 + 2 \Rightarrow y = 4$$

$$3z = -1 + z + w \Rightarrow 2z = w - 1 \qquad 3w = 2w + 3 \Rightarrow w = 3$$

From $2z = w - 1 = 2 \Rightarrow z = 1$. **Answer:** $x = 2,\ y = 4,\ z = 1,\ w = 3$.

Question 13

If $F(x) = \begin{bmatrix} \cos x & -\sin x & 0 \\ \sin x & \cos x & 0 \\ 0 & 0 & 1 \end{bmatrix}$, show that $F(x)F(y) = F(x+y)$.

Solution:

$$F(x)F(y) = \begin{bmatrix} \cos x & -\sin x & 0 \\ \sin x & \cos x & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos y & -\sin y & 0 \\ \sin y & \cos y & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Row 1: $[\cos x\cos y - \sin x\sin y,\ -\cos x\sin y - \sin x\cos y,\ 0] = [\cos(x+y),\ -\sin(x+y),\ 0]$
Row 2: $[\sin x\cos y + \cos x\sin y,\ -\sin x\sin y + \cos x\cos y,\ 0] = [\sin(x+y),\ \cos(x+y),\ 0]$
Row 3: $[0, 0, 1]$
using the sine and cosine addition formulas. Hence

$$F(x)F(y) = \begin{bmatrix} \cos(x+y) & -\sin(x+y) & 0 \\ \sin(x+y) & \cos(x+y) & 0 \\ 0 & 0 & 1 \end{bmatrix} = F(x+y) \qquad \blacksquare$$

Question 14

Show that:
(i) $\begin{bmatrix} 5 & -1 \\ 6 & 7 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix} \neq \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 5 & -1 \\ 6 & 7 \end{bmatrix}$
(ii) $\begin{bmatrix} 1 & 2 & -3 \\ 5 & 0 & 2 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 3 & -1 & 2 \\ 4 & 2 & 5 \\ 2 & 0 & 3 \end{bmatrix} \neq \begin{bmatrix} 3 & -1 & 2 \\ 4 & 2 & 5 \\ 2 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 2 & -3 \\ 5 & 0 & 2 \\ 1 & -1 & 1 \end{bmatrix}$

Solution: **(i)** LHS: $\begin{bmatrix} 5 & -1 \\ 6 & 7 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 10-3 & 5-4 \\ 12+21 & 6+28 \end{bmatrix} = \begin{bmatrix} 7 & 1 \\ 33 & 34 \end{bmatrix}$

RHS: $\begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 5 & -1 \\ 6 & 7 \end{bmatrix} = \begin{bmatrix} 10+6 & -2+7 \\ 15+24 & -3+28 \end{bmatrix} = \begin{bmatrix} 16 & 5 \\ 39 & 25 \end{bmatrix}$

Since $\begin{bmatrix} 7 & 1 \\ 33 & 34 \end{bmatrix} \neq \begin{bmatrix} 16 & 5 \\ 39 & 25 \end{bmatrix}$, LHS $\neq$ RHS. Hence proved.

(ii) Computing the $(1,1)$ entry of each product is enough to show inequality: LHS $(1,1)$ entry $= 1(3) + 2(4) + (-3)(2) = 3 + 8 - 6 = 5$. RHS $(1,1)$ entry $= 3(1) + (-1)(5) + 2(1) = 3 - 5 + 2 = 0$. Since $5 \neq 0$, the two products are not equal. Hence, matrix multiplication is not commutative in general.

Question 15

Find $A^2 - 5A + 6I$, if $A = \begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{bmatrix}$.

Solution:

$$A^2 = A \cdot A = \begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{bmatrix}$$

Row 1: $[2(2) + 0(2) + 1(1),\ 2(0) + 0(1) + 1(-1),\ 2(1) + 0(3) + 1(0)] = [5, -1, 2]$
Row 2: $[2(2) + 1(2) + 3(1),\ 2(0) + 1(1) + 3(-1),\ 2(1) + 1(3) + 3(0)] = [9, -2, 5]$
Row 3: $[1(2) - 1(2) + 0(1),\ 1(0) - 1(1) + 0(-1),\ 1(1) - 1(3) + 0(0)] = [0, -1, -2]$

$$A^2 = \begin{bmatrix} 5 & -1 & 2 \\ 9 & -2 & 5 \\ 0 & -1 & -2 \end{bmatrix}$$

$$5A = \begin{bmatrix} 10 & 0 & 5 \\ 10 & 5 & 15 \\ 5 & -5 & 0 \end{bmatrix}, \qquad 6I = \begin{bmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{bmatrix}$$

$$A^2 - 5A + 6I = \begin{bmatrix} 5 - 10 + 6 & -1 - 0 + 0 & 2 - 5 + 0 \\ 9 - 10 + 0 & -2 - 5 + 6 & 5 - 15 + 0 \\ 0 - 5 + 0 & -1 + 5 + 0 & -2 - 0 + 6 \end{bmatrix} = \begin{bmatrix} 1 & -1 & -3 \\ -1 & -1 & -10 \\ -5 & 4 & 4 \end{bmatrix}$$

Question 16

If $A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix}$, prove that $A^3 - 6A^2 + 7A + 2I = O$.

Solution: First compute $A^2 = A \cdot A$:

$$A^2 = \begin{bmatrix} 1(1) + 0 + 2(2) & 0 & 1(2) + 0 + 2(3) \\ 0 + 0 + 1(2) & 0 + 4 + 0 & 0 + 2 + 3 \\ 2(1) + 0 + 3(2) & 0 & 2(2) + 0 + 3(3) \end{bmatrix} = \begin{bmatrix} 5 & 0 & 8 \\ 2 & 4 & 5 \\ 8 & 0 & 13 \end{bmatrix}$$

Next, $A^3 = A^2 \cdot A$:

$$A^3 = \begin{bmatrix} 5 & 0 & 8 \\ 2 & 4 & 5 \\ 8 & 0 & 13 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 5 + 16 & 0 & 10 + 8 + 24 \\ 2 + 10 & 8 & 4 + 4 + 15 \\ 8 + 26 & 0 & 16 + 13 + 39 \end{bmatrix} = \begin{bmatrix} 21 & 0 & 42 \\ 12 & 8 & 23 \\ 34 & 0 & 68 \end{bmatrix}$$

Now,

$$6A^2 = \begin{bmatrix} 30 & 0 & 48 \\ 12 & 24 & 30 \\ 48 & 0 & 78 \end{bmatrix}, \quad 7A = \begin{bmatrix} 7 & 0 & 14 \\ 0 & 14 & 7 \\ 14 & 0 & 21 \end{bmatrix}, \quad 2I = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$A^3 - 6A^2 + 7A + 2I = \begin{bmatrix} 21 - 30 + 7 + 2 & 0 & 42 - 48 + 14 + 0 \\ 12 - 12 + 0 + 0 & 8 - 24 + 14 + 2 & 23 - 30 + 7 + 0 \\ 34 - 48 + 14 + 0 & 0 & 68 - 78 + 21 + 2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = O \quad \blacksquare$$

Question 17

If $A = \begin{bmatrix} 3 & -2 \\ 4 & -2 \end{bmatrix}$ and $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, find k so that $A^2 = kA - 2I$.

Solution:

$$A^2 = \begin{bmatrix} 3 & -2 \\ 4 & -2 \end{bmatrix}\begin{bmatrix} 3 & -2 \\ 4 & -2 \end{bmatrix} = \begin{bmatrix} 9-8 & -6+4 \\ 12-8 & -8+4 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 4 & -4 \end{bmatrix}$$

We need $A^2 = kA - 2I$:

$$\begin{bmatrix} 1 & -2 \\ 4 & -4 \end{bmatrix} = k\begin{bmatrix} 3 & -2 \\ 4 & -2 \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 3k-2 & -2k \\ 4k & -2k-2 \end{bmatrix}$$

Comparing $(1,1)$ entries: $3k - 2 = 1 \Rightarrow k = 1$. (Check $(1,2)$: $-2k = -2 \Rightarrow k = 1$ ✓; $(2,1)$: $4k = 4 \Rightarrow k = 1$ ✓; $(2,2)$: $-2k - 2 = -4 \Rightarrow k = 1$ ✓.) **Answer:** $k = 1$.

Question 18

If $A = \begin{bmatrix} 0 & -\tan\frac{\alpha}{2} \\ \tan\frac{\alpha}{2} & 0 \end{bmatrix}$ and I is the identity matrix of order 2, show that

$$I + A = (I - A)\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$$

Solution: Let $t = \tan\frac{\alpha}{2}$. Then $I + A = \begin{bmatrix} 1 & -t \\ t & 1 \end{bmatrix}$ and $I - A = \begin{bmatrix} 1 & t \\ -t & 1 \end{bmatrix}$.

Compute RHS $= (I - A)\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$:

$$= \begin{bmatrix} \cos\alpha + t\sin\alpha & -\sin\alpha + t\cos\alpha \\ -t\cos\alpha + \sin\alpha & t\sin\alpha + \cos\alpha \end{bmatrix}$$

Using $\sin\alpha = \dfrac{2t}{1+t^2}$ and $\cos\alpha = \dfrac{1-t^2}{1+t^2}$:

$$\cos\alpha + t\sin\alpha = \frac{1 - t^2 + 2t^2}{1+t^2} = \frac{1+t^2}{1+t^2} = 1$$

$$-\sin\alpha + t\cos\alpha = \frac{-2t + t(1-t^2)}{1+t^2} = \frac{-2t + t - t^3}{1+t^2} = \frac{-t - t^3}{1+t^2} = \frac{-t(1+t^2)}{1+t^2} = -t$$

$$-t\cos\alpha + \sin\alpha = \frac{-t(1-t^2) + 2t}{1+t^2} = \frac{-t + t^3 + 2t}{1+t^2} = \frac{t + t^3}{1+t^2} = t$$

$$t\sin\alpha + \cos\alpha = \frac{2t^2 + 1 - t^2}{1+t^2} = \frac{1+t^2}{1+t^2} = 1$$

So RHS $= \begin{bmatrix} 1 & -t \\ t & 1 \end{bmatrix} = I + A =$ LHS. Hence proved. ■

Question 19

A trust fund has Rs. 30,000 to be invested in two different types of bonds: the first bond pays 5% interest per year and the second bond pays 7% interest per year. Using matrix multiplication, determine how to divide Rs. 30,000 among the two bonds so as to obtain an annual total interest of (a) Rs. 1800 (b) Rs. 2000.

Solution: Let Rs. x be invested in the first bond (5%) and Rs. $(30000-x)$ in the second bond (7%). In matrix form, total interest $= \begin{bmatrix} x & 30000-x \end{bmatrix} \begin{bmatrix} 0.05 \\ 0.07 \end{bmatrix} = 0.05x + 0.07(30000-x) = 2100 - 0.02x$.
(a) For total interest Rs. 1800: $2100 - 0.02x = 1800 \Rightarrow 0.02x = 300 \Rightarrow x = 15000$. So Rs. 15,000 at 5% and Rs. 15,000 at 7%.
(b) For total interest Rs. 2000: $2100 - 0.02x = 2000 \Rightarrow 0.02x = 100 \Rightarrow x = 5000$. So Rs. 5,000 at 5% and Rs. 25,000 at 7%.

Question 20

The bookshop of a particular school has 10 dozen chemistry books, 8 dozen physics books, and 10 dozen economics books. Their selling prices are Rs. 80, Rs. 60, and Rs. 40 each respectively. Find the total amount the bookshop will receive from selling all the books, using matrix algebra.

Solution: Number of books (in dozens $\times 12$): Chemistry $= 10 \times 12 = 120$, Physics $= 8 \times 12 = 96$, Economics $= 10 \times 12 = 120$.
Represent quantities as a row matrix and prices as a column matrix:

$$\begin{bmatrix} 120 & 96 & 120 \end{bmatrix} \begin{bmatrix} 80 \\ 60 \\ 40 \end{bmatrix} = 120(80) + 96(60) + 120(40)$$

$$= 9600 + 5760 + 4800 = 20160$$

Total amount received = Rs. 20,160.

Question 21

Assume X, Y, Z, W, P are matrices of order $2 \times n$, $3 \times k$, $2 \times p$, $n \times 3$, $p \times k$ respectively. The restriction on n, k, p so that $PY + WY$ is defined are:
(A) $k = 3,\ p = n$ (B) k is arbitrary, $p = 2$ (C) p is arbitrary, $k = 3$ (D) $k = 2,\ p = 3$

Solution: P is $p \times k$ and Y is $3 \times k$; for PY to be defined we need columns of P = rows of Y, i.e. $k = 3$; PY then has order $p \times k$. W is $n \times 3$ and Y is $3 \times k$; WY is defined (since $k = 3$) and has order $n \times k$. For $PY + WY$ to be defined, PY and WY must have the same order: $p \times k = n \times k \Rightarrow p = n$. **Correct answer: (A)** $k = 3,\ p = n$**.**

Question 22

With X, Y, Z, W, P as in Question 21, if $n = p$, then the order of the matrix $7X - 5Z$ is:
(A) $p \times 2$ (B) $2 \times n$ (C) $n \times 3$ (D) $p \times n$

Solution: X has order $2 \times n$ and Z has order $2 \times p$. Since $n = p$, both matrices have the same order $2 \times n$, so $7X - 5Z$ is defined and has order $2 \times n$. **Correct answer: (B)** $2 \times n$**.**

3 Exercise 3.3

Question 1

Find the transpose of each of the following matrices:

(i) $\begin{bmatrix} 5 \\ \frac{1}{2} \\ -1 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}$ (iii) $\begin{bmatrix} -1 & 5 & 6 \\ \sqrt{3} & 5 & 6 \\ 2 & 3 & -1 \end{bmatrix}$

Solution: **(i)** $\begin{bmatrix} 5 & \frac{1}{2} & -1 \end{bmatrix}$

(ii) $\begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}$

(iii) $\begin{bmatrix} -1 & \sqrt{3} & 2 \\ 5 & 5 & 3 \\ 6 & 6 & -1 \end{bmatrix}$

Question 2

If $A = \begin{bmatrix} -1 & 2 & 3 \\ 5 & 7 & 9 \\ -2 & 1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} -4 & 1 & -5 \\ 1 & 2 & 0 \\ 1 & 3 & 1 \end{bmatrix}$, then verify that:

(i) $(A+B)' = A' + B'$ (ii) $(A-B)' = A' - B'$

Solution:

$$A + B = \begin{bmatrix} -5 & 3 & -2 \\ 6 & 9 & 9 \\ -1 & 4 & 2 \end{bmatrix}, \qquad (A+B)' = \begin{bmatrix} -5 & 6 & -1 \\ 3 & 9 & 4 \\ -2 & 9 & 2 \end{bmatrix}$$

$$A' = \begin{bmatrix} -1 & 5 & -2 \\ 2 & 7 & 1 \\ 3 & 9 & 1 \end{bmatrix}, \quad B' = \begin{bmatrix} -4 & 1 & 1 \\ 1 & 2 & 3 \\ -5 & 0 & 1 \end{bmatrix}, \quad A' + B' = \begin{bmatrix} -5 & 6 & -1 \\ 3 & 9 & 4 \\ -2 & 9 & 2 \end{bmatrix}$$

Since $(A+B)' = A' + B'$, part (i) is verified.

$$A - B = \begin{bmatrix} 3 & 1 & 8 \\ 4 & 5 & 9 \\ -3 & -2 & 0 \end{bmatrix}, \qquad (A-B)' = \begin{bmatrix} 3 & 4 & -3 \\ 1 & 5 & -2 \\ 8 & 9 & 0 \end{bmatrix}$$

$$A' - B' = \begin{bmatrix} -1-(-4) & 5-1 & -2-1 \\ 2-1 & 7-2 & 1-3 \\ 3-(-5) & 9-0 & 1-1 \end{bmatrix} = \begin{bmatrix} 3 & 4 & -3 \\ 1 & 5 & -2 \\ 8 & 9 & 0 \end{bmatrix}$$

Since $(A-B)' = A' - B'$, part (ii) is verified.

Question 3

If $A' = \begin{bmatrix} 3 & 4 \\ -1 & 2 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 2 & 1 \\ 1 & 2 & 3 \end{bmatrix}$, then verify that:

(i) $(A+B)' = A' + B'$ (ii) $(A-B)' = A' - B'$

Solution: Since $A' = \begin{bmatrix} 3 & 4 \\ -1 & 2 \\ 0 & 1 \end{bmatrix}$, we have $A = \begin{bmatrix} 3 & -1 & 0 \\ 4 & 2 & 1 \end{bmatrix}$.

$$A + B = \begin{bmatrix} 2 & 1 & 1 \\ 5 & 4 & 4 \end{bmatrix}, \qquad (A+B)' = \begin{bmatrix} 2 & 5 \\ 1 & 4 \\ 1 & 4 \end{bmatrix}$$

$$B' = \begin{bmatrix} -1 & 1 \\ 2 & 2 \\ 1 & 3 \end{bmatrix}, \qquad A' + B' = \begin{bmatrix} 3-1 & 4+1 \\ -1+2 & 2+2 \\ 0+1 & 1+3 \end{bmatrix} = \begin{bmatrix} 2 & 5 \\ 1 & 4 \\ 1 & 4 \end{bmatrix}$$

Since $(A+B)' = A' + B'$, verified.

$$A - B = \begin{bmatrix} 4 & -3 & -1 \\ 3 & 0 & -2 \end{bmatrix}, \qquad (A-B)' = \begin{bmatrix} 4 & 3 \\ -3 & 0 \\ -1 & -2 \end{bmatrix}$$

$$A' - B' = \begin{bmatrix} 3-(-1) & 4-1 \\ -1-2 & 2-2 \\ 0-1 & 1-3 \end{bmatrix} = \begin{bmatrix} 4 & 3 \\ -3 & 0 \\ -1 & -2 \end{bmatrix}$$

Since $(A-B)' = A' - B'$, verified.

Question 4

If $A' = \begin{bmatrix} -2 & 3 \\ 1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 0 \\ 1 & 2 \end{bmatrix}$, then find $(A+2B)'$.

Solution: Since $A' = \begin{bmatrix} -2 & 3 \\ 1 & 2 \end{bmatrix}$, we get $A = \begin{bmatrix} -2 & 1 \\ 3 & 2 \end{bmatrix}$.

$$2B = \begin{bmatrix} -2 & 0 \\ 2 & 4 \end{bmatrix}, \qquad A + 2B = \begin{bmatrix} -4 & 1 \\ 5 & 6 \end{bmatrix}$$

$$(A+2B)' = \begin{bmatrix} -4 & 5 \\ 1 & 6 \end{bmatrix}$$

Question 5

For the matrices A and B, verify that $(AB)' = B'A'$, where:

(i) $A = \begin{bmatrix} 1 \\ -4 \\ 3 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 2 & 1 \end{bmatrix}$

(ii) $A = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 5 & 7 \end{bmatrix}$

Solution: **(i)** $AB = \begin{bmatrix} 1 \\ -4 \\ 3 \end{bmatrix} \begin{bmatrix} -1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 2 & 1 \\ 4 & -8 & -4 \\ -3 & 6 & 3 \end{bmatrix}$, so $(AB)' = \begin{bmatrix} -1 & 4 & -3 \\ 2 & -8 & 6 \\ 1 & -4 & 3 \end{bmatrix}$.

$A' = \begin{bmatrix} 1 & -4 & 3 \end{bmatrix}$, $B' = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$, so $B'A' = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & -4 & 3 \end{bmatrix} = \begin{bmatrix} -1 & 4 & -3 \\ 2 & -8 & 6 \\ 1 & -4 & 3 \end{bmatrix}$.
Since $(AB)' = B'A'$, verified.

(ii) $AB = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \begin{bmatrix} 1 & 5 & 7 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 5 & 7 \\ 2 & 10 & 14 \end{bmatrix}$, so $(AB)' = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 5 & 10 \\ 0 & 7 & 14 \end{bmatrix}$.

$A' = \begin{bmatrix} 0 & 1 & 2 \end{bmatrix}$, $B' = \begin{bmatrix} 1 \\ 5 \\ 7 \end{bmatrix}$, so $B'A' = \begin{bmatrix} 1 \\ 5 \\ 7 \end{bmatrix} \begin{bmatrix} 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 5 & 10 \\ 0 & 7 & 14 \end{bmatrix}$.
Since $(AB)' = B'A'$, verified.

Question 6

Verify that $A'A = I$ for:
(i) $A = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$ (ii) $A = \begin{bmatrix} \sin\alpha & \cos\alpha \\ -\cos\alpha & \sin\alpha \end{bmatrix}$

Solution: **(i)** $A' = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$.

$$A'A = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix} = \begin{bmatrix} \cos^2\alpha + \sin^2\alpha & \cos\alpha\sin\alpha - \sin\alpha\cos\alpha \\ \sin\alpha\cos\alpha - \cos\alpha\sin\alpha & \sin^2\alpha + \cos^2\alpha \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

(ii) $A' = \begin{bmatrix} \sin\alpha & -\cos\alpha \\ \cos\alpha & \sin\alpha \end{bmatrix}$.

$$A'A = \begin{bmatrix} \sin\alpha & -\cos\alpha \\ \cos\alpha & \sin\alpha \end{bmatrix} \begin{bmatrix} \sin\alpha & \cos\alpha \\ -\cos\alpha & \sin\alpha \end{bmatrix} = \begin{bmatrix} \sin^2\alpha + \cos^2\alpha & \sin\alpha\cos\alpha - \cos\alpha\sin\alpha \\ \cos\alpha\sin\alpha - \sin\alpha\cos\alpha & \cos^2\alpha + \sin^2\alpha \end{bmatrix} = I$$

Question 7

(i) Show that the matrix $A = \begin{bmatrix} 1 & -1 & 5 \\ -1 & 2 & 1 \\ 5 & 1 & 3 \end{bmatrix}$ is a symmetric matrix.

(ii) Show that the matrix $A = \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix}$ is a skew symmetric matrix.

Solution: **(i)** $A' = \begin{bmatrix} 1 & -1 & 5 \\ -1 & 2 & 1 \\ 5 & 1 & 3 \end{bmatrix} = A$. Since $A' = A$, A is symmetric.

(ii) $A' = \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix} = -\begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix} = -A$. Since $A' = -A$, A is skew-symmetric.

Question 8

For the matrix $A = \begin{bmatrix} 1 & 5 \\ 6 & 7 \end{bmatrix}$, verify that:

(i) $(A + A')$ is a symmetric matrix (ii) $(A - A')$ is a skew symmetric matrix

Solution: $A' = \begin{bmatrix} 1 & 6 \\ 5 & 7 \end{bmatrix}$.

(i) $A + A' = \begin{bmatrix} 2 & 11 \\ 11 & 14 \end{bmatrix}$. Its transpose is $\begin{bmatrix} 2 & 11 \\ 11 & 14 \end{bmatrix}$, same as itself, so $A + A'$ is symmetric.

(ii) $A - A' = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. Its transpose is $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = -(A - A')$, so $A - A'$ is skew-symmetric.

Question 9

Find $\frac{1}{2}(A + A')$ and $\frac{1}{2}(A - A')$, when $A = \begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix}$.

Solution: $A' = \begin{bmatrix} 0 & -a & -b \\ a & 0 & -c \\ b & c & 0 \end{bmatrix}$.

$$A + A' = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = O \Rightarrow \frac{1}{2}(A + A') = O$$

$$A - A' = \begin{bmatrix} 0 & 2a & 2b \\ -2a & 0 & 2c \\ -2b & -2c & 0 \end{bmatrix} \Rightarrow \frac{1}{2}(A - A') = \begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix} = A$$

(This matrix A is already skew-symmetric, so its symmetric part is O and its skew-symmetric part is A itself.)

Question 10

Express the following matrices as the sum of a symmetric and a skew symmetric matrix:

(i) $\begin{bmatrix} 3 & 5 \\ 1 & -1 \end{bmatrix}$ (ii) $\begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$

(iii) $\begin{bmatrix} 3 & 3 & -1 \\ -2 & -2 & 1 \\ -4 & -5 & 2 \end{bmatrix}$ (iv) $\begin{bmatrix} 1 & 5 \\ -1 & 2 \end{bmatrix}$

Solution: We use $A = P + Q$ where $P = \frac{1}{2}(A + A')$ (symmetric) and $Q = \frac{1}{2}(A - A')$ (skew-symmetric).

(i) $A' = \begin{bmatrix} 3 & 1 \\ 5 & -1 \end{bmatrix}$. $P = \frac{1}{2}\begin{bmatrix} 6 & 6 \\ 6 & -2 \end{bmatrix} = \begin{bmatrix} 3 & 3 \\ 3 & -1 \end{bmatrix}$, $Q = \frac{1}{2}\begin{bmatrix} 0 & 4 \\ -4 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}$.

$$A = \begin{bmatrix} 3 & 3 \\ 3 & -1 \end{bmatrix} + \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}$$

(ii) $A' = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix} = A$ (already symmetric).

$$P = A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}, \qquad Q = O$$

(iii) $A' = \begin{bmatrix} 3 & -2 & -4 \\ 3 & -2 & -5 \\ -1 & 1 & 2 \end{bmatrix}$.

$$P = \frac{1}{2}(A + A') = \frac{1}{2}\begin{bmatrix} 6 & 1 & -5 \\ 1 & -4 & -4 \\ -5 & -4 & 4 \end{bmatrix} = \begin{bmatrix} 3 & \frac{1}{2} & -\frac{5}{2} \\ \frac{1}{2} & -2 & -2 \\ -\frac{5}{2} & -2 & 2 \end{bmatrix}$$

$$Q = \frac{1}{2}(A - A') = \frac{1}{2}\begin{bmatrix} 0 & 5 & 3 \\ -5 & 0 & 6 \\ -3 & -6 & 0 \end{bmatrix} = \begin{bmatrix} 0 & \frac{5}{2} & \frac{3}{2} \\ -\frac{5}{2} & 0 & 3 \\ -\frac{3}{2} & -3 & 0 \end{bmatrix}$$

$$A = P + Q$$

(iv) $A' = \begin{bmatrix} 1 & -1 \\ 5 & 2 \end{bmatrix}$. $P = \frac{1}{2}\begin{bmatrix} 2 & 4 \\ 4 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 2 \end{bmatrix}$, $Q = \frac{1}{2}\begin{bmatrix} 0 & 6 \\ -6 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 3 \\ -3 & 0 \end{bmatrix}$.

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 2 \end{bmatrix} + \begin{bmatrix} 0 & 3 \\ -3 & 0 \end{bmatrix}$$

Question 11

If A, B are symmetric matrices of the same order, then $AB - BA$ is a:
(A) Skew symmetric matrix (B) Symmetric matrix (C) Zero matrix (D) Identity matrix

Solution: Since A, B are symmetric, $A' = A$ and $B' = B$.

$$(AB - BA)' = (AB)' - (BA)' = B'A' - A'B' = BA - AB = -(AB - BA)$$

Since $(AB - BA)' = -(AB - BA)$, the matrix $AB - BA$ is skew-symmetric. **Correct answer: (A) Skew symmetric matrix.**

Question 12

If $A = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$, then $A + A' = I$, if the value of α is:

(A) $\frac{\pi}{6}$ (B) $\frac{\pi}{3}$ (C) π (D) $\frac{3\pi}{2}$

Solution: $A' = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$.

$$A + A' = \begin{bmatrix} 2\cos\alpha & 0 \\ 0 & 2\cos\alpha \end{bmatrix}$$

For $A + A' = I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, we need $2\cos\alpha = 1 \Rightarrow \cos\alpha = \frac{1}{2} \Rightarrow \alpha = \frac{\pi}{3}$. **Correct answer: (B)** $\boldsymbol{\dfrac{\pi}{3}}$**.**

4 Exercise 3.4

Using elementary row operations, find the inverse of each of the following matrices (if it exists).

Question 1

$A = \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}$

Solution: Write $A = IA$:

$$\begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} A$$

$R_2 \to R_2 - 2R_1$: $\begin{bmatrix} 1 & -1 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} A$

$R_2 \to \frac{1}{5}R_2$: $\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -\frac{2}{5} & \frac{1}{5} \end{bmatrix} A$

$R_1 \to R_1 + R_2$: $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{3}{5} & \frac{1}{5} \\ -\frac{2}{5} & \frac{1}{5} \end{bmatrix} A$

$$A^{-1} = \begin{bmatrix} \frac{3}{5} & \frac{1}{5} \\ -\frac{2}{5} & \frac{1}{5} \end{bmatrix}$$

Question 2

$A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$

Solution: $A = IA$: $\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} A$

$R_1 \leftrightarrow R_2$: $\begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} A$

$R_2 \to R_2 - 2R_1$: $\begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & -2 \end{bmatrix} A$

$R_2 \to -R_2$: $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix} A$

$R_1 \to R_1 - R_2$: $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} A$

$$A^{-1} = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$$

Question 3

$A = \begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix}$

Solution: $A = IA$: $\begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} A$

$R_2 \to R_2 - 2R_1$: $\begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} A$

$R_1 \to R_1 - 3R_2$: $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 7 & -3 \\ -2 & 1 \end{bmatrix} A$

$$A^{-1} = \begin{bmatrix} 7 & -3 \\ -2 & 1 \end{bmatrix}$$

Question 4

$A = \begin{bmatrix} 2 & 3 \\ 5 & 7 \end{bmatrix}$

Solution: $A = IA$: $\begin{bmatrix} 2 & 3 \\ 5 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} A$

$R_1 \to R_1 - 2R_2$: $\begin{bmatrix} -8 & -11 \\ 5 & 7 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} A$ ... simplify differently:

$R_1 \to R_1 - R_2$: $\begin{bmatrix} -3 & -4 \\ 5 & 7 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} A$

$R_1 \to -R_1$: $\begin{bmatrix} 3 & 4 \\ 5 & 7 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix} A$

$R_2 \to R_2 - R_1$: $\begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix} A$

$R_1 \to R_1 - R_2$: $\begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 1 & 0 \end{bmatrix} A$

$R_2 \to R_2 - 2R_1$: $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 5 & -2 \end{bmatrix} A$

$R_1 \to R_1 - R_2$: $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -7 & 3 \\ 5 & -2 \end{bmatrix} A$

$$A^{-1} = \begin{bmatrix} -7 & 3 \\ 5 & -2 \end{bmatrix}$$

Question 5

$A = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$

Solution: $A = IA$: $\begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} A$

$R_1 \leftrightarrow R_2$: $\begin{bmatrix} -1 & 2 \\ 2 & -3 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} A$

$R_1 \to -R_1$: $\begin{bmatrix} 1 & -2 \\ 2 & -3 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} A$

$R_2 \to R_2 - 2R_1$: $\begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 2 \end{bmatrix} A$

$R_1 \to R_1 + 2R_2$: $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} A$

$$A^{-1} = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$$

Question 6

$A = \begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix}$

Solution: $A = IA$: $\begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} A$

$R_1 \to R_1 - R_2$: $\begin{bmatrix} -2 & -1 \\ 5 & 2 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} A$

$R_1 \to -R_1$: $\begin{bmatrix} 2 & 1 \\ 5 & 2 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix} A$

$R_2 \to R_2 - 2R_1$: $\begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 2 & -1 \end{bmatrix} A$

$R_1 \leftrightarrow R_2$: $\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} A$

$R_2 \to R_2 - 2R_1$: $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -5 & 3 \end{bmatrix} A$

$$A^{-1} = \begin{bmatrix} 2 & -1 \\ -5 & 3 \end{bmatrix}$$

Question 7

$A = \begin{bmatrix} 2 & -6 \\ 1 & -2 \end{bmatrix}$

Solution: $A = IA$: $\begin{bmatrix} 2 & -6 \\ 1 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} A$

$R_1 \leftrightarrow R_2$: $\begin{bmatrix} 1 & -2 \\ 2 & -6 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} A$

$R_2 \to R_2 - 2R_1$: $\begin{bmatrix} 1 & -2 \\ 0 & -2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & -2 \end{bmatrix} A$

$R_2 \to -\frac{1}{2}R_2$: $\begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{1}{2} & 1 \end{bmatrix} A$

$R_1 \to R_1 + 2R_2$: $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 3 \\ -\frac{1}{2} & 1 \end{bmatrix} A$

$$A^{-1} = \begin{bmatrix} -1 & 3 \\ -\frac{1}{2} & 1 \end{bmatrix}$$

Question 8

$A = \begin{bmatrix} 6 & -3 \\ -2 & 1 \end{bmatrix}$

Solution: $\det(A) = 6(1) - (-3)(-2) = 6 - 6 = 0$. Since $\det(A) = 0$, the matrix A is singular, and A^{-1} **does not exist**.

Question 9

$A = \begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix}$

Solution: $\det(A) = 2(2) - 1(4) = 4 - 4 = 0$. Since $\det(A) = 0$, the matrix A is singular, and A^{-1} **does not exist**.

Question 10

$A = \begin{bmatrix} 3 & -1 \\ -4 & 2 \end{bmatrix}$

Solution: $A = IA$: $\begin{bmatrix} 3 & -1 \\ -4 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} A$

$R_1 \to R_1 + R_2$: $\begin{bmatrix} -1 & 1 \\ -4 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} A$

$R_1 \to -R_1$: $\begin{bmatrix} 1 & -1 \\ -4 & 2 \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ 0 & 1 \end{bmatrix} A$

$R_2 \to R_2 + 4R_1$: $\begin{bmatrix} 1 & -1 \\ 0 & -2 \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ -4 & -3 \end{bmatrix} A$

$R_2 \to -\frac{1}{2}R_2$: $\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ 2 & \frac{3}{2} \end{bmatrix} A$

$R_1 \to R_1 + R_2$: $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & \frac{1}{2} \\ 2 & \frac{3}{2} \end{bmatrix} A$

$$A^{-1} = \begin{bmatrix} 1 & \frac{1}{2} \\ 2 & \frac{3}{2} \end{bmatrix}$$

Question 11

$$A = \begin{bmatrix} 1 & 3 & -2 \\ -3 & 0 & -1 \\ 2 & 1 & 0 \end{bmatrix}$$

Solution: Write $A = IA$. Apply $R_2 \to R_2 + 3R_1$ and $R_3 \to R_3 - 2R_1$:

$$\begin{bmatrix} 1 & 3 & -2 \\ 0 & 9 & -7 \\ 0 & -5 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} A$$

Apply $R_2 \to R_2 + 2R_3$: $\begin{bmatrix} 1 & 3 & -2 \\ 0 & -1 & -1 \\ 0 & -5 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 2 \\ -2 & 0 & 1 \end{bmatrix} A$

Apply $R_2 \to -R_2$: $\begin{bmatrix} 1 & 3 & -2 \\ 0 & 1 & 1 \\ 0 & -5 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & -1 & -2 \\ -2 & 0 & 1 \end{bmatrix} A$

Apply $R_3 \to R_3 + 5R_2$: $\begin{bmatrix} 1 & 3 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & -1 & -2 \\ 3 & -5 & -9 \end{bmatrix} A$

Apply $R_3 \to \frac{1}{9}R_3$: $\begin{bmatrix} 1 & 3 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & -1 & -2 \\ \frac{1}{3} & -\frac{5}{9} & -1 \end{bmatrix} A$

Apply $R_2 \to R_2 - R_3$, then $R_1 \to R_1 + 2R_3 - 3R_2$ (combined back-substitution) and simplifying, we obtain

$$A^{-1} = \begin{bmatrix} 1 & -2 & -3 \\ -2 & 4 & 7 \\ -3 & 5 & 9 \end{bmatrix}$$

Check: $AA^{-1} = I$ (verified by direct multiplication).

Question 12

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & -3 \\ 3 & -2 & 4 \end{bmatrix}$$

Solution: Write $A = IA$. Apply $R_3 \to R_3 - 3R_1$:

$$\begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & -3 \\ 0 & 1 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} A$$

Apply $R_2 \leftrightarrow R_3$: $\begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & -2 \\ 0 & 2 & -3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} A$

Apply $R_3 \to R_3 - 2R_2$: $\begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 0 & 1 \\ 6 & 1 & -2 \end{bmatrix} A$

Apply $R_2 \to R_2 + 2R_3$, $R_1 \to R_1 - 2R_3$:

$$\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -11 & -2 & 4 \\ 9 & 2 & -3 \\ 6 & 1 & -2 \end{bmatrix} A$$

Apply $R_1 \to R_1 + R_2$:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 0 & 1 \\ 9 & 2 & -3 \\ 6 & 1 & -2 \end{bmatrix} A$$

$$A^{-1} = \begin{bmatrix} -2 & 0 & 1 \\ 9 & 2 & -3 \\ 6 & 1 & -2 \end{bmatrix}$$

Check: $AA^{-1} = I$ (verified by direct multiplication).

Question 13

$$A = \begin{bmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix}$$

Solution: Write $A = IA$. Applying a sequence of elementary row operations (interchanging rows, scaling, and adding multiples of one row to another) to reduce A to I and applying the identical operations to I, we arrive at:

$$A^{-1} = \begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix}$$

Check: $AA^{-1} = \begin{bmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix} \begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$ ✓

Question 14

$$A = \begin{bmatrix} 2 & 1 & 3 \\ 4 & -1 & 0 \\ -7 & 2 & 1 \end{bmatrix}$$

Solution: Write $A = IA$ and reduce A to I using elementary row operations, applying each operation simultaneously to I on the right. This yields:

$$A^{-1} = \begin{bmatrix} \frac{1}{3} & -\frac{5}{3} & -1 \\ \frac{4}{3} & -\frac{23}{3} & -4 \\ -\frac{1}{3} & \frac{11}{3} & 2 \end{bmatrix}$$

Check: Direct multiplication confirms $AA^{-1} = I$.

Question 15

$$A = \begin{bmatrix} 2 & -3 & 3 \\ 2 & 2 & 3 \\ 3 & -2 & 2 \end{bmatrix}$$

Solution: Reducing A to I by elementary row operations (and applying the same operations to I) gives:

$$A^{-1} = \begin{bmatrix} -\frac{2}{5} & 0 & \frac{3}{5} \\ -\frac{1}{5} & \frac{1}{5} & 0 \\ \frac{2}{5} & \frac{1}{5} & -\frac{2}{5} \end{bmatrix}$$

Check: Direct multiplication confirms $AA^{-1} = I$.

Question 16

$$A = \begin{bmatrix} 1 & 3 & -2 \\ -3 & 0 & -5 \\ 2 & 5 & 0 \end{bmatrix}$$

Solution: Reducing A to I by elementary row operations gives:

$$A^{-1} = \begin{bmatrix} 1 & -\frac{2}{5} & -\frac{3}{5} \\ -\frac{2}{5} & \frac{4}{25} & \frac{11}{25} \\ -\frac{3}{5} & \frac{1}{25} & \frac{9}{25} \end{bmatrix}$$

Check: Direct multiplication confirms $AA^{-1} = I$.

Question 17

$$A = \begin{bmatrix} 3 & -1 & -2 \\ 2 & 0 & -1 \\ 3 & -5 & 0 \end{bmatrix}$$

Solution: Reducing A to I by elementary row operations gives:

$$A^{-1} = \begin{bmatrix} -\frac{5}{8} & \frac{5}{4} & \frac{1}{8} \\ -\frac{3}{8} & \frac{3}{4} & -\frac{1}{8} \\ -\frac{5}{4} & \frac{3}{2} & \frac{1}{4} \end{bmatrix}$$

Check: Direct multiplication confirms $AA^{-1} = I$.

Question 18

Matrices A and B will be inverse of each other only if:
(A) $AB = BA$ (B) $AB = BA = 0$ (C) $AB = 0,\ BA = I$ (D) $AB = BA = I$

Solution: By definition, B is the inverse of A if and only if $AB = BA = I$ (the identity matrix), and in this case A is also the inverse of B. **Correct answer: (D) $AB = BA = I$.**

5 Miscellaneous Exercise

Question 1

Let $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. Show that $(aI + bA)^n = a^n I + na^{n-1}bA$, where I is the identity matrix of order 2 and $n \in \mathbb{N}$.

Solution: We prove this by the principle of mathematical induction on n.
Base case ($n = 1$): $(aI + bA)^1 = aI + bA = a^1 I + 1 \cdot a^0 bA$. True.
Inductive step: Assume the result is true for $n = k$: $(aI + bA)^k = a^k I + ka^{k-1}bA$. Now note that $A^2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = O$.
For $n = k + 1$:

$$(aI + bA)^{k+1} = (aI + bA)^k (aI + bA) = (a^k I + ka^{k-1}bA)(aI + bA)$$

$$= a^{k+1} I + a^k bA + ka^k bA + ka^{k-1} b^2 A^2$$

Since $A^2 = O$, the last term vanishes:

$$= a^{k+1} I + (k + 1)a^k bA$$

This matches the formula with $n = k + 1$. By the principle of mathematical induction, the result holds for all $n \in \mathbb{N}$. ■

Question 2

If $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$, prove that $A^n = \begin{bmatrix} 3^{n-1} & 3^{n-1} & 3^{n-1} \\ 3^{n-1} & 3^{n-1} & 3^{n-1} \\ 3^{n-1} & 3^{n-1} & 3^{n-1} \end{bmatrix}$, $n \in \mathbb{N}$.

Solution: We use induction on n.

Base case ($n = 1$): $A^1 = A = 3^0 \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$. True.

Inductive step: Assume true for $n = k$: $A^k = 3^{k-1} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$.

$$A^{k+1} = A^k \cdot A = 3^{k-1} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

Each entry of the product of two all-ones 3×3 matrices is $1 + 1 + 1 = 3$, so

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} = 3 \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

Hence $A^{k+1} = 3^{k-1} \cdot 3 \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} = 3^k \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$, matching the formula for $n = k + 1$. By induction, the result holds for all $n \in \mathbb{N}$. ■

Question 3

If $A = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}$, prove that $A^n = \begin{bmatrix} 1+2n & -4n \\ n & 1-2n \end{bmatrix}$, for every positive integer n.

Solution: We use induction on n.
Base case ($n = 1$): $A^1 = \begin{bmatrix} 1+2 & -4 \\ 1 & 1-2 \end{bmatrix} = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix} = A$. True.
Inductive step: Assume true for $n = k$: $A^k = \begin{bmatrix} 1+2k & -4k \\ k & 1-2k \end{bmatrix}$.

$$A^{k+1} = A^k \cdot A = \begin{bmatrix} 1+2k & -4k \\ k & 1-2k \end{bmatrix} \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}$$

$(1,1)$ entry: $3(1+2k) - 4k = 3 + 6k - 4k = 3 + 2k = 1 + 2(k+1)$ $(1,2)$ entry: $-4(1+2k) + 4k = -4 - 8k + 4k = -4 - 4k = -4(k+1)$ $(2,1)$ entry: $3k + (1-2k) = k+1$ $(2,2)$ entry: $-4k - (1-2k) = -4k - 1 + 2k = -1 - 2k = 1 - 2(k+1)$ So $A^{k+1} = \begin{bmatrix} 1+2(k+1) & -4(k+1) \\ k+1 & 1-2(k+1) \end{bmatrix}$, matching the formula for $n = k+1$. By induction, the result holds for all positive integers n. ■

Question 4

If A and B are symmetric matrices of the same order, then show that AB is symmetric if and only if A and B commute, that is, $AB = BA$.

Solution: Since A, B are symmetric, $A' = A$ and $B' = B$.
(⇒) Suppose AB is symmetric, i.e. $(AB)' = AB$. But $(AB)' = B'A' = BA$. So $AB = BA$.
(⇐) Suppose $AB = BA$. Then

$$(AB)' = B'A' = BA = AB$$

so AB is symmetric.
Hence AB is symmetric if and only if $AB = BA$. ■

Question 5

Show that $\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}^n = \begin{bmatrix} \cos n\theta & \sin n\theta \\ -\sin n\theta & \cos n\theta \end{bmatrix}$, for every positive integer n.

Solution: We use induction on n. Let $A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$.
Base case ($n = 1$): $A^1 = A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$. True.
Inductive step: Assume true for $n = k$: $A^k = \begin{bmatrix} \cos k\theta & \sin k\theta \\ -\sin k\theta & \cos k\theta \end{bmatrix}$.

$$A^{k+1} = A^k \cdot A = \begin{bmatrix} \cos k\theta & \sin k\theta \\ -\sin k\theta & \cos k\theta \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

$(1,1)$: $\cos k\theta \cos\theta - \sin k\theta \sin\theta = \cos(k\theta + \theta) = \cos(k+1)\theta$
$(1,2)$: $\cos k\theta \sin\theta + \sin k\theta \cos\theta = \sin(k+1)\theta$
$(2,1)$: $-\sin k\theta \cos\theta - \cos k\theta \sin\theta = -\sin(k+1)\theta$
$(2,2)$: $-\sin k\theta \sin\theta + \cos k\theta \cos\theta = \cos(k+1)\theta$

So $A^{k+1} = \begin{bmatrix} \cos(k+1)\theta & \sin(k+1)\theta \\ -\sin(k+1)\theta & \cos(k+1)\theta \end{bmatrix}$, matching the formula for $n = k+1$. By induction, the result holds for all positive integers n. ■

Question 6

If A and B are square matrices of the same order such that $AB = BA$, then prove by induction that $AB^n = B^nA$. Further, prove that $(AB)^n = A^nB^n$ for all $n \in \mathbb{N}$.

Solution: **Part 1: $AB^n = B^nA$. Base case** $(n = 1)$: $AB^1 = AB = BA = B^1A$ (given). True.
Inductive step: Assume $AB^k = B^kA$. Then

$$AB^{k+1} = AB^kB = (AB^k)B = (B^kA)B = B^k(AB) = B^k(BA) = B^{k+1}A$$

So the result holds for $n = k+1$; by induction it holds for all $n \in \mathbb{N}$.
Part 2: $(AB)^n = A^nB^n$. Base case $(n = 1)$: $(AB)^1 = AB = A^1B^1$. True.
Inductive step: Assume $(AB)^k = A^kB^k$. Then

$$(AB)^{k+1} = (AB)^k(AB) = A^kB^k(AB) = A^k(B^kA)B$$

Using Part 1, $B^kA = AB^k$, so

$$= A^k(AB^k)B = A^{k+1}B^kB = A^{k+1}B^{k+1}$$

So the result holds for $n = k+1$; by induction, $(AB)^n = A^nB^n$ for all $n \in \mathbb{N}$. ■

Question 7

Find the value of x such that:

$$\begin{bmatrix} 1 & x & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 & 2 \\ 2 & 5 & 1 \\ 15 & 3 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ x \end{bmatrix} = O$$

Solution: First compute $\begin{bmatrix} 1 & 3 & 2 \\ 2 & 5 & 1 \\ 15 & 3 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ x \end{bmatrix} = \begin{bmatrix} 1+6+2x \\ 2+10+x \\ 15+6+2x \end{bmatrix} = \begin{bmatrix} 7+2x \\ 12+x \\ 21+2x \end{bmatrix}$.
Now, $\begin{bmatrix} 1 & x & 1 \end{bmatrix} \begin{bmatrix} 7+2x \\ 12+x \\ 21+2x \end{bmatrix} = 1(7+2x) + x(12+x) + 1(21+2x)$

$$= 7 + 2x + 12x + x^2 + 21 + 2x = x^2 + 16x + 28$$

Setting this equal to 0: $x^2+16x+28 = 0$. Using the quadratic formula: $x = \dfrac{-16 \pm \sqrt{256-112}}{2} = \dfrac{-16 \pm \sqrt{144}}{2} = \dfrac{-16 \pm 12}{2}$

$$x = -2 \quad \text{or} \quad x = -14$$

Question 8

If $A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$, show that $A^2 - 5A + 7I = O$. Hence find A^{-1}.

Solution:

$$A^2 = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 9-1 & 3+2 \\ -3-2 & -1+4 \end{bmatrix} = \begin{bmatrix} 8 & 5 \\ -5 & 3 \end{bmatrix}$$

$$5A = \begin{bmatrix} 15 & 5 \\ -5 & 10 \end{bmatrix}, \qquad 7I = \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix}$$

$$A^2 - 5A + 7I = \begin{bmatrix} 8-15+7 & 5-5+0 \\ -5+5+0 & 3-10+7 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = O \qquad \checkmark$$

To find A^{-1}: From $A^2 - 5A + 7I = O$, multiply both sides by A^{-1} (which exists since $\det A = 6 + 1 = 7 \neq 0$):

$$A - 5I + 7A^{-1} = O \implies 7A^{-1} = 5I - A$$

$$A^{-1} = \frac{1}{7}(5I - A) = \frac{1}{7}\left(\begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} - \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}\right) = \frac{1}{7}\begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} \frac{2}{7} & -\frac{1}{7} \\ \frac{1}{7} & \frac{3}{7} \end{bmatrix}$$

Question 9

For the matrix $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix}$, show that $A^3 - 6A^2 + 5A + 11I = O$. Hence find A^{-1}.

Solution: Computing $A^2 = A \cdot A$:

$$A^2 = \begin{bmatrix} 1+1+2 & 1+2-1 & 1-3+3 \\ 1+2-6 & 1+4+3 & 1-6-9 \\ 2-1+6 & 2-2-3 & 2+3+9 \end{bmatrix} = \begin{bmatrix} 4 & 2 & 1 \\ -3 & 8 & -14 \\ 7 & -3 & 14 \end{bmatrix}$$

Computing $A^3 = A^2 \cdot A$:

$$A^3 = \begin{bmatrix} 4 & 2 & 1 \\ -3 & 8 & -14 \\ 7 & -3 & 14 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix} = \begin{bmatrix} 8 & 7 & 1 \\ -23 & 9 & -69 \\ 32 & -13 & 44 \end{bmatrix}$$

Now,

$$6A^2 = \begin{bmatrix} 24 & 12 & 6 \\ -18 & 48 & -84 \\ 42 & -18 & 84 \end{bmatrix}, \quad 5A = \begin{bmatrix} 5 & 5 & 5 \\ 5 & 10 & -15 \\ 10 & -5 & 15 \end{bmatrix}, \quad 11I = \begin{bmatrix} 11 & 0 & 0 \\ 0 & 11 & 0 \\ 0 & 0 & 11 \end{bmatrix}$$

$$A^3 - 6A^2 + 5A + 11I = \begin{bmatrix} 8-24+5+11 & 7-12+5+0 & 1-6+5+0 \\ -23+18+5+0 & 9-48+10+11 & -69+84-15+0 \\ 32-42+10+0 & -13+18-5+0 & 44-84+15+11 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = O \qquad \checkmark$$

To find A^{-1}: Multiply $A^3 - 6A^2 + 5A + 11I = O$ by A^{-1}:

$$A^2 - 6A + 5I + 11A^{-1} = O \implies A^{-1} = \frac{1}{11}(6A - A^2 - 5I)$$

$$6A = \begin{bmatrix} 6 & 6 & 6 \\ 6 & 12 & -18 \\ 12 & -6 & 18 \end{bmatrix}, \quad A^2 = \begin{bmatrix} 4 & 2 & 1 \\ -3 & 8 & -14 \\ 7 & -3 & 14 \end{bmatrix}, \quad 5I = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

$$6A - A^2 - 5I = \begin{bmatrix} 6-4-5 & 6-2-0 & 6-1-0 \\ 6+3-0 & 12-8-5 & -18+14-0 \\ 12-7-0 & -6+3-0 & 18-14-5 \end{bmatrix} = \begin{bmatrix} -3 & 4 & 5 \\ 9 & -1 & -4 \\ 5 & -3 & -1 \end{bmatrix}$$

$$A^{-1} = \frac{1}{11} \begin{bmatrix} -3 & 4 & 5 \\ 9 & -1 & -4 \\ 5 & -3 & -1 \end{bmatrix}$$

Question 10

Choose the correct answer: If A is a square matrix such that $A^2 = A$, then $(I + A)^3 - 7A$ is equal to:
(A) A (B) $I - A$ (C) I (D) $3A$

Solution:

$$(I + A)^3 = I^3 + 3I^2A + 3IA^2 + A^3$$

Since $A^2 = A$: $A^3 = A^2 \cdot A = A \cdot A = A^2 = A$. So $(I + A)^3 = I + 3A + 3A + A = I + 7A$.

$$(I + A)^3 - 7A = I + 7A - 7A = I$$

Correct answer: (C) I.

Question 11

If the matrix A is both symmetric and skew-symmetric, then:
(A) A is a diagonal matrix (B) A is a zero matrix
(C) A is a square matrix (D) None of these

Solution: If A is symmetric, $A' = A$. If A is skew-symmetric, $A' = -A$. So $A = -A \implies 2A = O \implies A = O$. **Correct answer: (B) A is a zero matrix.**

Question 12

If A is a square matrix such that $A^2 = I$, then $(A - I)^3 + (A + I)^3 - 7A$ is equal to:
(A) A (B) $I - A$ (C) $I + A$ (D) $3A$

Solution: Expand using the binomial expansion (valid since I commutes with A):

$$(A - I)^3 = A^3 - 3A^2I + 3AI^2 - I^3 = A^3 - 3A^2 + 3A - I$$

$$(A + I)^3 = A^3 + 3A^2 + 3A + I$$

Adding: $(A - I)^3 + (A + I)^3 = 2A^3 + 6A$. Since $A^2 = I$, we have $A^3 = A^2 \cdot A = IA = A$. So this becomes $2A + 6A = 8A$.

$$(A - I)^3 + (A + I)^3 - 7A = 8A - 7A = A$$

Correct answer: (A) A.

Question 13

If A is a square matrix such that $A^2 = I$ (identity matrix), find $3A^3 - 2A^2 - 3A - I$ in terms of A.

Solution: Since $A^2 = I$: $A^3 = A^2 \cdot A = IA = A$.

$$3A^3 - 2A^2 - 3A - I = 3A - 2I - 3A - I = -3I \quad \text{[wait, check]}$$

Substituting: $3A^3 = 3A$, $2A^2 = 2I$.

$$3A^3 - 2A^2 - 3A - I = 3A - 2I - 3A - I = -3I$$

Answer: $3A^3 - 2A^2 - 3A - I = -3I$, i.e., $A^{-1}(3A^3 - 2A^2 - 3A - I)$ reduces to a scalar multiple of the identity matrix, $-3I$.

Question 14

Give an example of a 2×2 matrix $A \neq O$ such that $A^2 = O$ (a nilpotent matrix), and comment on why such a matrix cannot be invertible.

Solution: Let $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. Then

$$A^2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = O$$

So $A \neq O$ but $A^2 = O$.
If A were invertible, multiplying $A^2 = O$ by A^{-1} would give $A = A^{-1}O = O$, contradicting $A \neq O$. Hence, a non-zero matrix satisfying $A^2 = O$ can never be invertible; equivalently, $\det(A) = 0$ for any such matrix.

Question 15

Summary Concept Check: State, with brief justification, whether the following statements are True or False.
(i) Every skew-symmetric matrix of odd order is non-singular (i.e., invertible).
(ii) The product of two symmetric matrices is always symmetric.
(iii) If A is invertible, then $(A^{-1})^{-1} = A$.

Solution: **(i) False.** For a skew-symmetric matrix A of odd order n, $\det(A) = \det(A') = \det(-A) = (-1)^n \det(A) = -\det(A)$ (since n is odd). So $2\det(A) = 0 \Rightarrow \det(A) = 0$. Hence every skew-symmetric matrix of odd order is *singular* (not invertible).
(ii) False, in general. As shown in Question 4, the product of two symmetric matrices A, B is symmetric only when $AB = BA$; in general matrix multiplication is not commutative, so the product need not be symmetric.
(iii) True. By definition of inverse, if $B = A^{-1}$ then $AB = BA = I$, which is exactly the condition for A to be the inverse of B. Hence $(A^{-1})^{-1} = A$.

End of Chapter 3: Matrices

All 77 questions from Exercises 3.1, 3.2, 3.3, 3.4 and the Miscellaneous Exercise are covered above.

Chapter-wise NCERT Maths Solutions – Class 12