NCERT Class 12 Maths

Chapter-wise Complete Solutions

Chapter 6

Application of Derivatives

Complete Step-by-Step NCERT Solutions

Rate of Change • Increasing/Decreasing Functions • Tangents & Normals
Approximations • Maxima & Minima • Miscellaneous Exercise

What's inside:

- 70+ fully worked questions across all 5 exercises + Miscellaneous
- Key formulas & method summary before each exercise
- Every optimization problem solved with clear reasoning (first/second derivative tests)
- Clean, exam-ready mathematical notation throughout
- Common-mistake tips and a quick-revision formula sheet at the end

Prepared for Class 12 CBSE / NCERT students
Suitable for board-exam preparation and quick revision

Contents

Chapter Overview & Key Formulas

This chapter uses the derivative $\dfrac{dy}{dx}$ as a tool to solve real, practical problems: how fast a quantity is changing, where a function is going up or down, the equation of a line touching a curve, quick approximations, and finding the biggest or smallest possible value of something (optimization).

1. Rate of Change

If $y = f(x)$, then $\dfrac{dy}{dx}$ (or $\dfrac{dy}{dt}$ when both x, y depend on time t) represents the rate of change of y with respect to x (or t). Related-rate problems use the **chain rule**: $\dfrac{dV}{dt} = \dfrac{dV}{dr} \cdot \dfrac{dr}{dt}$.

2. Increasing / Decreasing Functions

Let f be differentiable on an interval I.

- f is **increasing** on I if $f'(x) \geq 0$ for all $x \in I$ (strictly increasing if $f'(x) > 0$).
- f is **decreasing** on I if $f'(x) \leq 0$ for all $x \in I$ (strictly decreasing if $f'(x) < 0$).

Method: find $f'(x)$, solve $f'(x) = 0$ to get critical points, then test the sign of $f'(x)$ in each sub-interval.

3. Tangents and Normals

At point (x_0, y_0) on curve $y = f(x)$, slope of tangent $m = \left.\dfrac{dy}{dx}\right|_{(x_0,y_0)}$.

$$\text{Tangent: } y - y_0 = m(x - x_0) \qquad \text{Normal: } y - y_0 = -\frac{1}{m}(x - x_0)\ (m \neq 0)$$

If $m = 0$: tangent is $y = y_0$ (horizontal), normal is $x = x_0$ (vertical). If tangent is vertical, normal is horizontal ($y = y_0$).

4. Approximations (Differentials)

For small Δx: $\quad \Delta y \approx dy = f'(x)\,\Delta x, \qquad f(x + \Delta x) \approx f(x) + f'(x)\Delta x$

5. Maxima and Minima

First Derivative Test: at a critical point c (where $f'(c) = 0$):

- f' changes $+ \to -$ at $c \Rightarrow$ local maximum
- f' changes $- \to +$ at $c \Rightarrow$ local minimum
- no sign change $\Rightarrow$ neither (point of inflection)

Second Derivative Test: at critical point c with $f'(c) = 0$:

- $f''(c) < 0 \Rightarrow$ local maximum; $\quad f''(c) > 0 \Rightarrow$ local minimum; $\quad f''(c) = 0 \Rightarrow$ test fails, use first derivative test

Absolute max/min on $[a, b]$*:* evaluate f at all critical points in (a, b) *and* at the endpoints a, b; the largest/smallest of these values is the absolute max/min.

Tip: In optimization word problems, always express the quantity to be maximised/minimised as a function of a *single* variable using the given constraint, differentiate, find critical points, and confirm maximum vs. minimum before stating the final answer.

1 Exercise 6.1: Rate of Change of Quantities

Question 1: Find the rate of change of the area of a circle with respect to its radius r when (a) $r = 3$ cm (b) $r = 4$ cm.

Solution: Area $A = \pi r^2$. Rate of change: $\frac{dA}{dr} = 2\pi r$.

(a) At $r = 3$: $\frac{dA}{dr} = 2\pi(3) = 6\pi\ \text{cm}^2/\text{cm}$.

(b) At $r = 4$: $\frac{dA}{dr} = 2\pi(4) = 8\pi\ \text{cm}^2/\text{cm}$.

Question 2: The volume of a cube is increasing at the rate of 8 cm^3/s. How fast is the surface area increasing when the length of an edge is 12 cm?

Solution: Let edge $= x$. $V = x^3 \Rightarrow \frac{dV}{dt} = 3x^2\frac{dx}{dt}$. Given $\frac{dV}{dt} = 8$:

$$8 = 3x^2\frac{dx}{dt} \Rightarrow \frac{dx}{dt} = \frac{8}{3x^2}$$

Surface area $S = 6x^2 \Rightarrow \frac{dS}{dt} = 12x\frac{dx}{dt} = 12x \cdot \frac{8}{3x^2} = \frac{32}{x}$.

At $x = 12$: $\frac{dS}{dt} = \frac{32}{12} = \frac{8}{3}\ \text{cm}^2/\text{s}$.

Question 3: The radius of a circle is increasing uniformly at the rate of 3 cm/s. Find the rate at which the area of the circle is increasing when the radius is 10 cm.

Solution: $A = \pi r^2 \Rightarrow \frac{dA}{dt} = 2\pi r\frac{dr}{dt}$. Given $\frac{dr}{dt} = 3$, at $r = 10$:

$$\frac{dA}{dt} = 2\pi(10)(3) = 60\pi\ \text{cm}^2/\text{s}$$

Question 4: An edge of a variable cube is increasing at the rate of 3 cm/s. How fast is the volume increasing when the edge is 10 cm long?

Solution: $V = x^3 \Rightarrow \dfrac{dV}{dt} = 3x^2\dfrac{dx}{dt}$. At $x = 10$, $\dfrac{dx}{dt} = 3$:

$$\frac{dV}{dt} = 3(100)(3) = 900 \text{ cm}^3/\text{s}$$

Question 5: The radius of a circle is increasing at the rate of 0.7 cm/s. What is the rate of increase of its circumference?

Solution: $C = 2\pi r \Rightarrow \dfrac{dC}{dt} = 2\pi\dfrac{dr}{dt} = 2\pi(0.7) = 1.4\pi$ cm/s

---

Question 6: The radius r of a sphere is increasing uniformly at the rate of 4 cm/s. How fast is the volume of the ball increasing when the radius is 10 cm?

Solution: $V = \dfrac{4}{3}\pi r^3 \Rightarrow \dfrac{dV}{dt} = 4\pi r^2\dfrac{dr}{dt}$. At $r = 10$, $\dfrac{dr}{dt} = 4$:

$$\frac{dV}{dt} = 4\pi(100)(4) = 1600\pi \text{ cm}^3/\text{s}$$

Question 7: A balloon, which always remains spherical, is being inflated by pumping in 900 cubic centimetres of gas per second. Find the rate at which the radius of the balloon increases when the radius is 15 cm.

Solution: $V = \dfrac{4}{3}\pi r^3 \Rightarrow \dfrac{dV}{dt} = 4\pi r^2\dfrac{dr}{dt}$. Given $\dfrac{dV}{dt} = 900$, at $r = 15$:

$$900 = 4\pi(225)\frac{dr}{dt} \quad\Rightarrow\quad \frac{dr}{dt} = \frac{900}{900\pi} = \frac{1}{\pi} \text{ cm/s}$$

Question 8: A balloon, which always remains spherical, has a variable diameter $\dfrac{3}{2}(2x+1)$. Find the rate of change of its volume with respect to x.

Solution: Radius $r = \dfrac{1}{2}\cdot\dfrac{3}{2}(2x+1) = \dfrac{3}{4}(2x+1)$.

$$V = \frac{4}{3}\pi r^3 = \frac{4}{3}\pi\left[\frac{3}{4}(2x+1)\right]^3 = \frac{4}{3}\pi\cdot\frac{27}{64}(2x+1)^3 = \frac{9\pi}{16}(2x+1)^3$$

$$\frac{dV}{dx} = \frac{9\pi}{16} \cdot 3(2x+1)^2 \cdot 2 = \frac{27\pi}{8}(2x+1)^2$$

Question 9: Sand is pouring from a pipe at the rate of 12 cm^3/s. The falling sand forms a cone whose height is always one-sixth of the radius of the base. How fast is the height of the sand cone increasing when the height is 4 cm?

Solution: Given $h = \frac{r}{6} \Rightarrow r = 6h$.

$$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi(6h)^2 h = 12\pi h^3 \;\Rightarrow\; \frac{dV}{dt} = 36\pi h^2 \frac{dh}{dt}$$

Given $\frac{dV}{dt} = 12$, at $h = 4$:

$$12 = 36\pi(16)\frac{dh}{dt} \;\Rightarrow\; \frac{dh}{dt} = \frac{12}{576\pi} = \frac{1}{48\pi} \text{ cm/s}$$

Question 10: The total cost $C(x)$ (in rupees) associated with production of x units of an item is $C(x) = 0.007x^3 - 0.003x^2 + 15x + 4000$. Find the marginal cost when 17 units are produced.

Solution: Marginal cost $= \frac{dC}{dx} = 0.021x^2 - 0.006x + 15$.
At $x = 17$:

$$MC = 0.021(289) - 0.006(17) + 15 = 6.069 - 0.102 + 15 = 20.967 \approx \text{Rs. } 20.97$$

Question 11: The total revenue in rupees received from the sale of x units of a product is given by $R(x) = 13x^2 + 26x + 15$. Find the marginal revenue when $x = 7$.

Solution: Marginal revenue $MR = \frac{dR}{dx} = 26x + 26$.
At $x = 7$: $MR = 26(7) + 26 = 182 + 26 = \text{Rs. } 208$

2 Exercise 6.2: Increasing and Decreasing Functions

Question 1: Show that the function $f(x) = 3x + 17$ is strictly increasing on $\mathbb{R}$.

Solution: $f'(x) = 3 > 0$ for all $x \in \mathbb{R}$. Since the derivative is always positive, f is strictly increasing on $\mathbb{R}$.

Question 2: Show that the function $f(x) = e^{2x}$ is strictly increasing on $\mathbb{R}$.

Solution: $f'(x) = 2e^{2x}$. Since $e^{2x} > 0$ for all x, $f'(x) > 0$ for all $x \in \mathbb{R}$. Hence f is strictly increasing on $\mathbb{R}$.

Question 3: Show that $f(x) = \sin x$ is (a) increasing in $\left(0, \frac{\pi}{2}\right)$, (b) decreasing in $\left(\frac{\pi}{2}, \pi\right)$, (c) neither increasing nor decreasing in $(0, \pi)$.

Solution: $f'(x) = \cos x$.

(a) On $\left(0, \frac{\pi}{2}\right)$: $\cos x > 0 \Rightarrow f'(x) > 0 \Rightarrow f$ increasing.

(b) On $\left(\frac{\pi}{2}, \pi\right)$: $\cos x < 0 \Rightarrow f'(x) < 0 \Rightarrow f$ decreasing.

(c) On $(0, \pi)$, $f'(x)$ is positive on part of the interval and negative on the rest, so f is neither increasing nor decreasing throughout $(0, \pi)$.

Question 4: Find the intervals in which $f(x) = 2x^2 - 3x$ is increasing or decreasing.

Solution: $f'(x) = 4x - 3$. Setting $f'(x) = 0 \Rightarrow x = \frac{3}{4}$.

For $x < \frac{3}{4}$: $f'(x) < 0 \Rightarrow$ decreasing. For $x > \frac{3}{4}$: $f'(x) > 0 \Rightarrow$ increasing.

$$\boxed{\text{Increasing on } \left(\tfrac{3}{4}, \infty\right); \quad \text{Decreasing on } \left(-\infty, \tfrac{3}{4}\right)}$$

Question 5: Find the intervals in which $f(x) = 2x^3 - 3x^2 - 36x + 7$ is increasing or decreasing.

Solution: $f'(x) = 6x^2 - 6x - 36 = 6(x^2 - x - 6) = 6(x-3)(x+2)$.
Critical points: $x = -2,\ 3$. Testing sign of $f'(x)$ in each interval:

$$\boxed{\text{Increasing on } (-\infty, -2) \cup (3, \infty); \quad \text{Decreasing on } (-2, 3)}$$

Question 6: Find the intervals in which $f(x) = 10 - 6x - 2x^2$ is increasing or decreasing.

Solution: $f'(x) = -6 - 4x$. Setting $f'(x) = 0 \Rightarrow x = -\dfrac{3}{2}$.
For $x < -\dfrac{3}{2}$: $f'(x) > 0 \Rightarrow$ increasing. For $x > -\dfrac{3}{2}$: $f'(x) < 0 \Rightarrow$ decreasing.

$$\boxed{\text{Increasing on } \left(-\infty, -\tfrac{3}{2}\right); \quad \text{Decreasing on } \left(-\tfrac{3}{2}, \infty\right)}$$

Question 7: Find the intervals in which $f(x) = 6 - 9x - x^2$ is increasing or decreasing.

Solution: $f'(x) = -9 - 2x = 0 \Rightarrow x = -\dfrac{9}{2}$.

$$\boxed{\text{Increasing on } \left(-\infty, -\tfrac{9}{2}\right); \quad \text{Decreasing on } \left(-\tfrac{9}{2}, \infty\right)}$$

Question 8: Find the intervals in which $f(x) = (x+1)^3(x-3)^3$ is increasing or decreasing.

Solution:

$$f'(x) = 3(x+1)^2(x-3)^3 + 3(x-3)^2(x+1)^3 = 3(x+1)^2(x-3)^2\left[(x-3)+(x+1)\right] = 6(x+1)^2(x-3)^2(x-1)$$

Since $(x+1)^2 \geq 0$ and $(x-3)^2 \geq 0$ always, the sign of $f'(x)$ is governed entirely by $(x-1)$.

$$\boxed{\text{Increasing on } (1, \infty); \quad \text{Decreasing on } (-\infty, 1)}$$

Question 9: Prove that $y = \big[x(x-2)\big]^2$ is increasing in $(0,1)\cup(2,\infty)$ and decreasing in $(-\infty,0)\cup(1,2)$.

Solution: $y = (x^2-2x)^2$. Let $u = x^2 - 2x$; then $y = u^2$ and

$$\frac{dy}{dx} = 2u\cdot u' = 2(x^2-2x)(2x-2) = 4x(x-2)(x-1)$$

Critical points: $x = 0, 1, 2$. Testing the sign of $x(x-2)(x-1)$ in each interval:

- $x < 0$: (neg)(neg)(neg) = negative $\Rightarrow$ decreasing
- $0 < x < 1$: (pos)(neg)(neg) = positive $\Rightarrow$ increasing
- $1 < x < 2$: (pos)(neg)(pos) = negative $\Rightarrow$ decreasing
- $x > 2$: (pos)(pos)(pos) = positive $\Rightarrow$ increasing

$$\boxed{\text{Increasing on } (0,1)\cup(2,\infty); \quad \text{Decreasing on } (-\infty,0)\cup(1,2)}$$

Hence proved. ■

Question 10: Find the intervals in which the function $f(x) = x^3 + \dfrac{1}{x^3}$ $(x \neq 0)$ is increasing or decreasing.

Solution:

$$f'(x) = 3x^2 - \frac{3}{x^4} = \frac{3x^6-3}{x^4} = \frac{3(x^6-1)}{x^4}$$

Since $x^4 > 0$ always (for $x \neq 0$), the sign of $f'(x)$ depends on $x^6 - 1$, i.e. on whether $|x| > 1$ or $|x| < 1$.

$$\boxed{\text{Increasing when } |x| > 1, \text{ i.e. on } (-\infty,-1)\cup(1,\infty); \quad \text{Decreasing when } 0 < |x| < 1}$$

Question 11: The function $f(x) = x^2e^{-x}$ is increasing in which interval?
(A) $(-\infty,\infty)$ (B) $(-2,0)$ (C) $(2,\infty)$ (D) $(0,2)$

Solution:

$$f'(x) = 2xe^{-x} - x^2e^{-x} = xe^{-x}(2-x)$$

Since $e^{-x} > 0$ always, sign of $f'(x)$ depends on $x(2-x)$, which is positive when $0 < x < 2$.

$$\boxed{\text{Option (D) is correct}}$$

Question 12: The interval in which $f(x) = x^{100} + \sin x - 1$ is decreasing is:
(A) $(1, 2)$ (B) $\left(\frac{\pi}{2}, \pi\right)$ (C) $(0, 1)$ (D) None of these

Solution: $f'(x) = 100x^{99} + \cos x$.

For $x > 0$, $100x^{99}$ grows very large while $\cos x \in [-1, 1]$, so $f'(x) > 0$ throughout each of the given intervals (checked directly, f' stays positive on all of $(1, 2)$, $\left(\frac{\pi}{2}, \pi\right)$ and $(0, 1)$). So f is increasing — never decreasing — on all three given options.

$$\boxed{\text{Option (D) is correct}}$$

3 Exercise 6.3: Tangents and Normals

Question 1: Find the slope of the tangent to the curve $y = 3x^4 - 4x$ at $x = 4$.

Solution: $\frac{dy}{dx} = 12x^3 - 4$. At $x = 4$: $12(64) - 4 = 768 - 4 = 764$.

Question 2: Find the slope of the tangent to the curve $y = \frac{x-1}{x-2}$, $x \neq 2$ at $x = 10$.

Solution:

$$\frac{dy}{dx} = \frac{(x-2)-(x-1)}{(x-2)^2} = \frac{-1}{(x-2)^2}$$

At $x = 10$: $\frac{dy}{dx} = \frac{-1}{64}$.

Question 3: Find the slope of the normal to the curve $x = a\cos^3\theta,\ y = a\sin^3\theta$ at $\theta = \frac{\pi}{4}$.

Solution: $\frac{dx}{d\theta} = -3a\cos^2\theta\sin\theta$, $\frac{dy}{d\theta} = 3a\sin^2\theta\cos\theta$.

$$\frac{dy}{dx} = \frac{3a\sin^2\theta\cos\theta}{-3a\cos^2\theta\sin\theta} = -\tan\theta$$

At $\theta = \frac{\pi}{4}$: slope of tangent $= -\tan\frac{\pi}{4} = -1$. Slope of normal $= -\frac{1}{-1} = 1$.

Question 4: Find the slope of the normal to the curve $x = 1 - a\sin\theta,\ y = b\cos^2\theta$ at $\theta = \frac{\pi}{2}$.

Solution: $\frac{dx}{d\theta} = -a\cos\theta$, $\frac{dy}{d\theta} = -2b\cos\theta\sin\theta$.

$$\frac{dy}{dx} = \frac{-2b\cos\theta\sin\theta}{-a\cos\theta} = \frac{2b\sin\theta}{a}$$

At $\theta = \frac{\pi}{2}$: slope of tangent $= \frac{2b}{a}$. Slope of normal $= -\frac{a}{2b}$.

Question 5: Find points on the curve $y = x^3 - 11x + 5$ at which the tangent is $y = x - 11$.

Solution: Slope of given line $= 1$. $\dfrac{dy}{dx} = 3x^2 - 11 = 1 \Rightarrow x^2 = 4 \Rightarrow x = \pm 2$.
At $x = 2$: $y = 8 - 22 + 5 = -9$. Check on line: $2 - 11 = -9$ ✓. Point $(2, -9)$ valid.
At $x = -2$: $y = -8 + 22 + 5 = 19$. Check on line: $-2 - 11 = -13 \neq 19$. Rejected.

$$\boxed{\text{The required point is } (2, -9)}$$

Question 6: Find the equations of all lines having slope -1 that are tangent to the curve $y = \dfrac{1}{x-1}$, $x \neq 1$.

Solution: $\dfrac{dy}{dx} = \dfrac{-1}{(x-1)^2} = -1 \Rightarrow (x-1)^2 = 1 \Rightarrow x = 2$ or $x = 0$.
At $x = 2$: $y = 1$, point $(2, 1)$. Tangent: $y - 1 = -1(x - 2) \Rightarrow x + y - 3 = 0$.
At $x = 0$: $y = -1$, point $(0, -1)$. Tangent: $y + 1 = -1(x - 0) \Rightarrow x + y + 1 = 0$.

$$\boxed{x + y - 3 = 0 \quad \text{and} \quad x + y + 1 = 0}$$

Question 7: Find points on the curve $y = x^3$ at which the slope of the tangent equals the y-coordinate of the point.

Solution: $\dfrac{dy}{dx} = 3x^2$. Setting this equal to $y = x^3$: $3x^2 = x^3 \Rightarrow x^2(3 - x) = 0 \Rightarrow x = 0$ or $x = 3$.
At $x = 0$: $y = 0$, point $(0, 0)$. At $x = 3$: $y = 27$, point $(3, 27)$.

$$\boxed{(0, 0) \text{ and } (3, 27)}$$

Question 8: For the curve $y = 4x^3 - 2x^5$, find the points at which the tangent passes through the origin.

Solution: At a point $(a, 4a^3 - 2a^5)$, slope $= 12a^2 - 10a^4$. Since the tangent must pass through the origin, this slope also equals $\dfrac{y - 0}{x - 0} = \dfrac{4a^3 - 2a^5}{a} = 4a^2 - 2a^4$ (for $a \neq 0$).

$$12a^2 - 10a^4 = 4a^2 - 2a^4 \;\Rightarrow\; 8a^2 - 8a^4 = 0 \;\Rightarrow\; 8a^2(1 - a^2) = 0 \;\Rightarrow\; a = 0, \pm 1$$

$a = 0 \Rightarrow (0, 0)$; $\quad a = 1 \Rightarrow (1, 2)$; $\quad a = -1 \Rightarrow (-1, -2)$.

$$\boxed{(0, 0),\ (1, 2),\ (-1, -2)}$$

Question 9: Find the points on the curve $x^2 + y^2 - 2x - 3 = 0$ at which the tangents are parallel to the x-axis.

Solution: Differentiating implicitly: $2x + 2y\dfrac{dy}{dx} - 2 = 0 \Rightarrow \dfrac{dy}{dx} = \dfrac{1-x}{y}$.

Tangent parallel to x-axis $\Rightarrow$ slope $= 0 \Rightarrow 1 - x = 0 \Rightarrow x = 1$.

Substituting in the curve: $1 + y^2 - 2 - 3 = 0 \Rightarrow y^2 = 4 \Rightarrow y = \pm 2$.

$$\boxed{(1, 2) \text{ and } (1, -2)}$$

Question 10: Find the equation of the tangent to $y = x^2 - 2x + 7$ which is (a) parallel to the line $2x - y + 9 = 0$, (b) perpendicular to the line $5y - 15x = 13$.

Solution: $\dfrac{dy}{dx} = 2x - 2$.

(a) Slope of given line $= 2$. Set $2x - 2 = 2 \Rightarrow x = 2,\ y = 4 - 4 + 7 = 7$. Point $(2, 7)$.

Tangent: $y - 7 = 2(x - 2) \Rightarrow \boxed{2x - y + 3 = 0}$

(b) Line $5y - 15x = 13 \Rightarrow y = 3x + \frac{13}{5}$, slope $= 3$; perpendicular slope $= -\dfrac{1}{3}$.

$$2x - 2 = -\frac{1}{3} \ \Rightarrow\ x = \frac{5}{6}, \qquad y = \left(\frac{5}{6}\right)^2 - 2\left(\frac{5}{6}\right) + 7 = \frac{217}{36}$$

Tangent: $y - \dfrac{217}{36} = -\dfrac{1}{3}\left(x - \dfrac{5}{6}\right)$. Multiplying through by 36:

$$\boxed{12x + 36y = 227}$$

Question 11: Find the equations of the tangent and normal to the curve $x^{2/3} + y^{2/3} = 2$ at $(1, 1)$.

Solution: Differentiating implicitly: $\dfrac{2}{3}x^{-1/3} + \dfrac{2}{3}y^{-1/3}\dfrac{dy}{dx} = 0 \ \Rightarrow\ \dfrac{dy}{dx} = -\left(\dfrac{y}{x}\right)^{1/3}$.

At $(1, 1)$: $\dfrac{dy}{dx} = -1$.

Tangent: $y - 1 = -1(x - 1) \Rightarrow \boxed{x + y = 2}$

Normal (slope = 1): $y - 1 = 1(x - 1) \Rightarrow \boxed{y = x}$

Question 12: Find the equation of the tangent to the curve $y = \sqrt{3x - 2}$ which is parallel to the line $4x - 2y + 5 = 0$.

Solution: $\dfrac{dy}{dx} = \dfrac{3}{2\sqrt{3x-2}}$. Line: $4x - 2y + 5 = 0 \Rightarrow y = 2x + \frac{5}{2}$, slope = 2.

$$\frac{3}{2\sqrt{3x-2}} = 2 \;\Rightarrow\; \sqrt{3x-2} = \frac{3}{4} \;\Rightarrow\; 3x - 2 = \frac{9}{16} \;\Rightarrow\; x = \frac{41}{48}$$

$$y = \sqrt{3\left(\frac{41}{48}\right) - 2} = \sqrt{\frac{9}{16}} = \frac{3}{4}$$

Tangent at $\left(\dfrac{41}{48}, \dfrac{3}{4}\right)$: $y - \dfrac{3}{4} = 2\left(x - \dfrac{41}{48}\right)$

Question 13: Prove that the curves $x = y^2$ and $xy = k$ cut at right angles if $8k^2 = 1$.

Solution: Curve 1 ($x = y^2$): $1 = 2y\dfrac{dy}{dx} \Rightarrow m_1 = \dfrac{1}{2y}$.

Curve 2 ($xy = k$): $y + x\dfrac{dy}{dx} = 0 \Rightarrow m_2 = -\dfrac{y}{x}$.

For orthogonal intersection, $m_1 m_2 = -1$:

$$\frac{1}{2y} \cdot \left(-\frac{y}{x}\right) = -\frac{1}{2x} = -1 \;\Rightarrow\; x = \frac{1}{2}$$

From $x = y^2$: $y^2 = \dfrac{1}{2} \Rightarrow y = \dfrac{1}{\sqrt{2}}$. From $xy = k$: $k = \dfrac{1}{2} \cdot \dfrac{1}{\sqrt{2}} = \dfrac{1}{2\sqrt{2}}$.

$$8k^2 = 8 \cdot \frac{1}{8} = 1$$

Hence proved. ■

Question 14: Find the angle of intersection of the curves $y^2 = x$ and $x^2 = y$.

Solution: Solving simultaneously: substitute $y = x^2$ into $y^2 = x$: $x^4 = x \Rightarrow x(x^3 - 1) = 0 \Rightarrow x = 0, 1$. Intersection points: $(0, 0)$ and $(1, 1)$.

At $(1,1)$**:** For $y^2 = x$: $2y\,dy/dx = 1 \Rightarrow m_1 = \dfrac{1}{2y} = \dfrac{1}{2}$. For $x^2 = y$: $dy/dx = 2x \Rightarrow m_2 = 2$.

$$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right| = \left|\frac{\frac{1}{2} - 2}{1 + 1}\right| = \frac{3}{4} \;\Rightarrow\; \theta = \tan^{-1}\left(\frac{3}{4}\right)$$

At $(0,0)$**:** curve $y^2 = x$ has a vertical tangent there, while $x^2 = y$ has a horizontal tangent there, so the curves meet at right angles: $\theta = \dfrac{\pi}{2}$.

---

4 Exercise 6.4: Approximations

Question 1: Using differentials, find the approximate value of $\sqrt{25.3}$.

Solution: Let $y = \sqrt{x}$, take $x = 25$, $\Delta x = 0.3$. $\frac{dy}{dx} = \frac{1}{2\sqrt{x}} = \frac{1}{10}$ at $x = 25$.

$$\Delta y \approx \frac{dy}{dx}\Delta x = \frac{1}{10}(0.3) = 0.03$$

$$\sqrt{25.3} \approx \sqrt{25} + 0.03 = 5 + 0.03 = \boxed{5.03}$$

Question 2: Using differentials, find the approximate value of $\sqrt{49.5}$.

Solution: $x = 49$, $\Delta x = 0.5$. $\frac{dy}{dx} = \frac{1}{2\sqrt{49}} = \frac{1}{14}$.

$$\Delta y \approx \frac{1}{14}(0.5) = 0.0357$$

$$\sqrt{49.5} \approx 7 + 0.0357 = \boxed{7.0357}$$

Question 3: Using differentials, find the approximate value of $\sqrt{0.6}$.

Solution: Take $x = 0.64$, $\Delta x = -0.04$ (since $\sqrt{0.64} = 0.8$ is easy). $\frac{dy}{dx} = \frac{1}{2\sqrt{0.64}} = \frac{1}{1.6} = 0.625$.

$$\Delta y \approx 0.625 \times (-0.04) = -0.025$$

$$\sqrt{0.6} \approx 0.8 - 0.025 = \boxed{0.775}$$

Question 4: Using differentials, find the approximate value of $(0.009)^{1/3}$.

Solution: Take $x = 0.008$, $\Delta x = 0.001$ (since $(0.008)^{1/3} = 0.2$). $y = x^{1/3} \Rightarrow \frac{dy}{dx} = \frac{1}{3}x^{-2/3} = \frac{1}{3}(0.04)^{-1} = 8.\overline{3}$.

$$\Delta y \approx 8.333 \times 0.001 = 0.008333$$

$$(0.009)^{1/3} \approx 0.2 + 0.0083 = \boxed{0.2083}$$

Question 5: Using differentials, find the approximate value of $(0.999)^{1/10}$.

Solution: Take $x = 1,\ \Delta x = -0.001$. $\dfrac{dy}{dx} = \dfrac{1}{10}x^{-9/10} = 0.1$ at $x = 1$.

$$\Delta y \approx 0.1 \times (-0.001) = -0.0001$$

$$(0.999)^{1/10} \approx 1 - 0.0001 = \boxed{0.9999}$$

Question 6: Using differentials, find the approximate value of $(15)^{1/4}$.

Solution: Take $x = 16,\ \Delta x = -1$. $\dfrac{dy}{dx} = \dfrac{1}{4}x^{-3/4} = \dfrac{1}{4} \cdot \dfrac{1}{8} = \dfrac{1}{32}$ at $x = 16$.

$$\Delta y \approx \frac{1}{32} \times (-1) = -0.03125$$

$$(15)^{1/4} \approx 2 - 0.03125 = \boxed{1.96875}$$

Question 7: Find the approximate value of $f(5.001)$, where $f(x) = x^3 - 7x^2 + 15$.

Solution: $f(5) = 125 - 175 + 15 = -35$. $f'(x) = 3x^2 - 14x \Rightarrow f'(5) = 75 - 70 = 5$.

$$f(5.001) \approx f(5) + f'(5) \times 0.001 = -35 + 5(0.001) = \boxed{-34.995}$$

Question 8: Find the approximate percentage error in calculating the volume of a spherical ball if an error of 2% is made in measuring its radius.

Solution: $V = \frac{4}{3}\pi r^3$. Taking logarithmic differentials: $\frac{dV}{V} = 3\frac{dr}{r}$.

Given $\frac{dr}{r} \times 100 = 2\%$:

$$\frac{dV}{V} \times 100 = 3 \times 2\% = \boxed{6\%}$$

Question 9: If the side of a cube is measured as x metres with a possible error of 1%, find the approximate error in calculating its volume.

Solution: $V = x^3 \Rightarrow dV = 3x^2\,dx$. Given $dx = 1\%$ of $x = 0.01x$.

$$dV = 3x^2(0.01x) = 0.03x^3 \text{ m}^3$$

So the approximate error in the calculated volume is $\boxed{0.03x^3 \text{ m}^3}$ (i.e. 3% of the volume).

5 Exercise 6.5: Maxima and Minima

Question 1: Find the local maximum and local minimum value of $f(x) = x^2$.

Solution: $f'(x) = 2x = 0 \Rightarrow x = 0$. $f''(x) = 2 > 0 \Rightarrow$ local minimum at $x = 0$.

$$\boxed{\text{Local minimum value} = f(0) = 0;\ \text{no local maximum}}$$

Question 2: Find the local maximum and minimum values of $f(x) = x^3 - 3x$.

Solution: $f'(x) = 3x^2 - 3 = 0 \Rightarrow x = \pm 1$. $f''(x) = 6x$.
At $x = 1$: $f''(1) = 6 > 0 \Rightarrow$ local minimum, $f(1) = 1 - 3 = -2$.
At $x = -1$: $f''(-1) = -6 < 0 \Rightarrow$ local maximum, $f(-1) = -1 + 3 = 2$.

$$\boxed{\text{Local max} = 2 \text{ at } x = -1; \quad \text{Local min} = -2 \text{ at } x = 1}$$

Question 3: Find the absolute maximum and minimum values of $f(x) = x^3$ on the interval $[-2, 2]$.

Solution: $f'(x) = 3x^2 \geq 0$ always, so f is increasing throughout; the only critical point is $x = 0$ (not a max/min). Checking endpoints:

$$f(-2) = -8, \qquad f(2) = 8$$

$$\boxed{\text{Absolute maximum} = 8 \text{ at } x = 2; \quad \text{Absolute minimum} = -8 \text{ at } x = -2}$$

Question 4: Find the absolute maximum and minimum values of $f(x) = 12x^{4/3} - 6x^{1/3}$ on $[-1, 1]$.

Solution:

$$f'(x) = 16x^{1/3} - 2x^{-2/3} = 2x^{-2/3}(8x - 1)$$

Critical points: $x = \dfrac{1}{8}$ (where $f' = 0$) and $x = 0$ (where f' is undefined but f is defined). Evaluate f at $-1, 0, \frac{1}{8}, 1$:

$$f(-1) = 12(1) - 6(-1) = 18, \quad f(0) = 0, \quad f\left(\tfrac{1}{8}\right) = 12\left(\tfrac{1}{16}\right) - 6\left(\tfrac{1}{2}\right) = -\tfrac{9}{4}, \quad f(1) = 12 - 6 = 6$$

$$\boxed{\text{Absolute max} = 18 \text{ at } x = -1; \quad \text{Absolute min} = -\tfrac{9}{4} \text{ at } x = \tfrac{1}{8}}$$

Question 5: Find two numbers whose sum is 24 and whose product is maximum.

Solution: Let the numbers be x and $24 - x$. Product $P(x) = x(24 - x) = 24x - x^2$.

$$P'(x) = 24 - 2x = 0 \Rightarrow x = 12, \qquad P''(x) = -2 < 0 \Rightarrow \text{maximum}$$

$$\boxed{\text{The numbers are 12 and 12; maximum product} = 144}$$

Question 6: Find two positive numbers x and y such that $x + y = 60$ and xy^3 is maximum.

Solution: $y = 60 - x$. $f(x) = x(60 - x)^3$.

$$f'(x) = (60 - x)^3 - 3x(60 - x)^2 = (60 - x)^2\big[(60 - x) - 3x\big] = (60 - x)^2(60 - 4x)$$

Critical points: $x = 60$ (rejected, gives zero) and $x = 15$. Sign of f' changes from $+$ to $-$ at $x = 15$, confirming a maximum.

$$\boxed{x = 15, \; y = 45}$$

Question 7: Show that all the rectangles with a given perimeter, the square has the maximum area.

Solution: Let perimeter $= P$ (fixed), sides x and $\dfrac{P}{2} - x$. Area $A(x) = x\left(\dfrac{P}{2} - x\right) = \dfrac{Px}{2} - x^2$.

$$A'(x) = \frac{P}{2} - 2x = 0 \;\Rightarrow\; x = \frac{P}{4}, \qquad A''(x) = -2 < 0 \text{ (maximum)}$$

Other side $= \dfrac{P}{2} - \dfrac{P}{4} = \dfrac{P}{4} = x$. Both sides equal, so the rectangle is a **square**. Hence proved. ■

Question 8: Show that of all the rectangles inscribed in a given circle, the square has the maximum area.

Solution: Let the circle have radius r; a rectangle inscribed in it has sides x, y with diagonal $= 2r$, i.e. $x^2 + y^2 = 4r^2$.

Area $A = xy = x\sqrt{4r^2 - x^2}$. Maximize $A^2 = x^2(4r^2 - x^2)$. Let $u = x^2$: $g(u) = u(4r^2 - u) = 4r^2u - u^2$.

$$g'(u) = 4r^2 - 2u = 0 \Rightarrow u = 2r^2 \Rightarrow x^2 = 2r^2 \Rightarrow x = r\sqrt{2}$$

Then $y^2 = 4r^2 - 2r^2 = 2r^2 \Rightarrow y = r\sqrt{2} = x$. Since $x = y$, the rectangle is a **square** (side $r\sqrt{2}$). Hence proved. ■

Question 9: Show that the right circular cylinder of given surface area and maximum volume is such that its height is equal to the diameter of the base.

Solution: Let radius r, height h, fixed surface area $S = 2\pi rh + 2\pi r^2 \Rightarrow h = \dfrac{S - 2\pi r^2}{2\pi r}$.

$$V = \pi r^2 h = \pi r^2 \cdot \frac{S - 2\pi r^2}{2\pi r} = \frac{Sr - 2\pi r^3}{2}$$

$$\frac{dV}{dr} = \frac{S - 6\pi r^2}{2} = 0 \;\Rightarrow\; S = 6\pi r^2$$

Substituting back: $h = \dfrac{6\pi r^2 - 2\pi r^2}{2\pi r} = \dfrac{4\pi r^2}{2\pi r} = 2r$.

Since $\dfrac{d^2V}{dr^2} = -6\pi r < 0$, this is indeed a maximum. So $h = 2r =$ diameter of base. Hence proved. ■

Question 10: Show that of all the rectangles of given area, the one with the least perimeter is a square.

Solution: Let area $= A$ (fixed), sides x, y with $xy = A \Rightarrow y = \dfrac{A}{x}$. Perimeter $P(x) = 2\left(x + \dfrac{A}{x}\right)$.

$$P'(x) = 2\left(1 - \frac{A}{x^2}\right) = 0 \;\Rightarrow\; x^2 = A \;\Rightarrow\; x = \sqrt{A}$$

Then $y = \dfrac{A}{\sqrt{A}} = \sqrt{A} = x$. Since $P''(x) = \dfrac{4A}{x^3} > 0$, this is a minimum. As $x = y$, the rectangle is a **square**. Hence proved. ■

Question 11: Show that the semi-vertical angle of the right circular cone of given surface area and maximum volume is $\sin^{-1}\left(\dfrac{1}{3}\right)$.

Solution: Let radius r, slant height l, total surface area (fixed) $S = \pi r l + \pi r^2 \Rightarrow l = \dfrac{S - \pi r^2}{\pi r}$.

Height $h = \sqrt{l^2 - r^2}$, and $V^2 = \dfrac{1}{9}\pi^2 r^4 (l^2 - r^2)$. Substituting for l:

$$l^2 - r^2 = \frac{(S - \pi r^2)^2 - \pi^2 r^4}{\pi^2 r^2} = \frac{(S - 2\pi r^2)(S)}{\pi^2 r^2} \quad \text{(difference of squares)}$$

$$V^2 = \frac{1}{9}\pi^2 r^4 \cdot \frac{S(S - 2\pi r^2)}{\pi^2 r^2} = \frac{S}{9}\left(S r^2 - 2\pi r^4\right)$$

Differentiating w.r.t. r and equating to zero:

$$\frac{d(V^2)}{dr} = \frac{S}{9}\left(2Sr - 8\pi r^3\right) = 0 \;\Rightarrow\; S = 4\pi r^2$$

Substituting back into $S = \pi r l + \pi r^2$: $4\pi r^2 = \pi r l + \pi r^2 \Rightarrow \pi r l = 3\pi r^2 \Rightarrow l = 3r$.

If θ is the semi-vertical angle, $\sin\theta = \dfrac{r}{l} = \dfrac{r}{3r} = \dfrac{1}{3} \;\Rightarrow\; \theta = \sin^{-1}\left(\dfrac{1}{3}\right)$.

(This can be confirmed to be a maximum by checking the sign of $\dfrac{d^2(V^2)}{dr^2}$ at this point.) Hence proved. ∎

Question 12: Show that the semi-vertical angle of the right circular cone of given volume and least curved surface area is $\cot^{-1}(\sqrt{2})$.

Solution: Let volume V (fixed) $= \dfrac{1}{3}\pi r^2 h \Rightarrow h = \dfrac{3V}{\pi r^2}$.

Curved surface area $A = \pi r l = \pi r\sqrt{r^2 + h^2}$. Consider $Z = A^2 = \pi^2 r^2 (r^2 + h^2) = \pi^2 r^4 + \pi^2 r^2 h^2$.

Substituting $h = \dfrac{3V}{\pi r^2}$: $\pi^2 r^2 h^2 = \pi^2 r^2 \cdot \dfrac{9V^2}{\pi^2 r^4} = \dfrac{9V^2}{r^2}$.

$$Z = \pi^2 r^4 + \frac{9V^2}{r^2}$$

$$\frac{dZ}{dr} = 4\pi^2 r^3 - \frac{18V^2}{r^3} = 0 \;\Rightarrow\; r^6 = \frac{18V^2}{4\pi^2} = \frac{9V^2}{2\pi^2}$$

Using $V = \dfrac{1}{3}\pi r^2 h \Rightarrow V^2 = \dfrac{\pi^2 r^4 h^2}{9}$, substitute:

$$r^6 = \frac{9}{2\pi^2} \cdot \frac{\pi^2 r^4 h^2}{9} = \frac{r^4 h^2}{2} \;\Rightarrow\; r^2 = \frac{h^2}{2} \;\Rightarrow\; h = r\sqrt{2}$$

If θ is the semi-vertical angle, $\tan\theta = \dfrac{r}{h} = \dfrac{r}{r\sqrt{2}} = \dfrac{1}{\sqrt{2}} \Rightarrow \theta = \cot^{-1}(\sqrt{2})$.

Hence proved. ■

Question 13: An open box with a square base is to be made out of a given quantity of cardboard of area c^2. Show that the maximum volume of the box is $\frac{c^3}{6\sqrt{3}}$.

Solution: Let the base side $= x$, height $= h$. Since the box is open at the top: $x^2 + 4xh = c^2 \Rightarrow h = \frac{c^2 - x^2}{4x}$.

$$V = x^2 h = x^2 \cdot \frac{c^2 - x^2}{4x} = \frac{c^2 x - x^3}{4}$$

$$\frac{dV}{dx} = \frac{c^2 - 3x^2}{4} = 0 \;\Rightarrow\; x^2 = \frac{c^2}{3} \;\Rightarrow\; x = \frac{c}{\sqrt{3}}$$

$$h = \frac{c^2 - \frac{c^2}{3}}{4 \cdot \frac{c}{\sqrt{3}}} = \frac{\frac{2c^2}{3}}{\frac{4c}{\sqrt{3}}} = \frac{2c^2\sqrt{3}}{12c} = \frac{c}{2\sqrt{3}}$$

$$V_{max} = x^2 h = \frac{c^2}{3} \cdot \frac{c}{2\sqrt{3}} = \frac{c^3}{6\sqrt{3}}$$

(Second derivative confirms this is a maximum.) Hence proved. ■

Question 14: A wire of length 28 m is cut into two pieces, one to make a square and the other a circle. Find the lengths of the pieces so that the combined area is minimum.

Solution: Let the piece of length x form the square (side $= x/4$) and the remaining piece $(28 - x)$ form the circle, so $2\pi r = 28 - x \Rightarrow r = \frac{28 - x}{2\pi}$.

$$A(x) = \left(\frac{x}{4}\right)^2 + \pi r^2 = \frac{x^2}{16} + \frac{(28 - x)^2}{4\pi}$$

$$A'(x) = \frac{x}{8} - \frac{28 - x}{2\pi} = 0 \;\Rightarrow\; \pi x = 4(28 - x) \;\Rightarrow\; x(\pi + 4) = 112 \;\Rightarrow\; x = \frac{112}{\pi + 4}$$

Remaining piece for circle: $28 - x = \frac{28(\pi + 4) - 112}{\pi + 4} = \frac{28\pi}{\pi + 4}$.

Since $A''(x) = \frac{1}{8} + \frac{1}{2\pi} > 0$, this is a minimum.

$$\boxed{\text{Square piece} = \frac{112}{\pi + 4}\text{ m}; \quad \text{Circle piece} = \frac{28\pi}{\pi + 4}\text{ m}}$$

Question 15: Prove that the volume of the largest cone that can be inscribed in a sphere of radius R is $\frac{8}{27}$ of the volume of the sphere.

Solution: Let the cone (inscribed in the sphere, apex on the sphere) have height h and base radius r. A standard geometric relation gives $r^2 = h(2R - h)$.

$$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi h^2(2R - h) = \frac{1}{3}\pi(2Rh^2 - h^3)$$

$$\frac{dV}{dh} = \frac{1}{3}\pi(4Rh - 3h^2) = 0 \;\Rightarrow\; h(4R - 3h) = 0 \;\Rightarrow\; h = \frac{4R}{3}\ (h \neq 0)$$

$$V_{max} = \frac{1}{3}\pi\left[2R\left(\frac{4R}{3}\right)^2 - \left(\frac{4R}{3}\right)^3\right] = \frac{1}{3}\pi\left[\frac{32R^3}{9} - \frac{64R^3}{27}\right] = \frac{1}{3}\pi \cdot \frac{32R^3}{27} = \frac{32\pi R^3}{81}$$

Volume of sphere $= \frac{4}{3}\pi R^3$. Ratio:

$$\frac{V_{max}}{\frac{4}{3}\pi R^3} = \frac{32\pi R^3/81}{4\pi R^3/3} = \frac{32}{81} \times \frac{3}{4} = \frac{8}{27}$$

Hence proved. ■

6 Miscellaneous Exercise on Chapter 6

Question 1: Show that the function $f(x) = 2x + \cot^{-1} x + \log\left(\sqrt{1+x^2} - x\right)$ is increasing on $\mathbb{R}$.

Solution:

$$f'(x) = 2 - \frac{1}{1+x^2} + \frac{d}{dx}\log\left(\sqrt{1+x^2} - x\right)$$

For the log term: $\frac{d}{dx}\log(\sqrt{1+x^2} - x) = \frac{\frac{x}{\sqrt{1+x^2}} - 1}{\sqrt{1+x^2} - x} = \frac{-\left(\sqrt{1+x^2} - x\right)/\sqrt{1+x^2}}{\sqrt{1+x^2} - x} = -\frac{1}{\sqrt{1+x^2}}$

$$f'(x) = 2 - \frac{1}{1+x^2} - \frac{1}{\sqrt{1+x^2}}$$

Since $\frac{1}{1+x^2} \le 1$ and $\frac{1}{\sqrt{1+x^2}} \le 1$ for all x (with equality only at $x = 0$),

$$f'(x) \ge 2 - 1 - 1 = 0$$

with $f'(x) > 0$ for $x \ne 0$. Hence f is increasing on $\mathbb{R}$. Hence proved. ■

Question 2: Show that $y = \log(1+x) - \frac{2x}{2+x}$, $x > -1$, is an increasing function of x throughout its domain.

Solution:

$$f'(x) = \frac{1}{1+x} - \frac{2(2+x) - 2x}{(2+x)^2} = \frac{1}{1+x} - \frac{4}{(2+x)^2}$$

$$= \frac{(2+x)^2 - 4(1+x)}{(1+x)(2+x)^2} = \frac{4 + 4x + x^2 - 4 - 4x}{(1+x)(2+x)^2} = \frac{x^2}{(1+x)(2+x)^2}$$

For $x > -1$: $(1+x) > 0$, $(2+x)^2 > 0$, and $x^2 \ge 0$. Hence $f'(x) \ge 0$ throughout the domain, so f is increasing. Hence proved. ■

Question 3: Show that $y = \frac{4\sin\theta}{2+\cos\theta} - \theta$ is an increasing function of θ in $\left[0, \frac{\pi}{2}\right]$.

Solution:

$$f'(\theta) = \frac{4\cos\theta(2+\cos\theta) - 4\sin\theta(-\sin\theta)}{(2+\cos\theta)^2} - 1 = \frac{8\cos\theta + 4\cos^2\theta + 4\sin^2\theta}{(2+\cos\theta)^2} - 1 = \frac{8\cos\theta + 4}{(2+\cos\theta)^2} - 1$$

$$= \frac{8\cos\theta + 4 - (2+\cos\theta)^2}{(2+\cos\theta)^2} = \frac{8\cos\theta + 4 - 4 - 4\cos\theta - \cos^2\theta}{(2+\cos\theta)^2} = \frac{\cos\theta(4 - \cos\theta)}{(2+\cos\theta)^2}$$

For $\theta \in \left[0, \dfrac{\pi}{2}\right]$: $\cos\theta \geq 0$ and $4 - \cos\theta > 0$ always. So $f'(\theta) \geq 0$, hence f is increasing. Hence proved. ■

Question 4: Find the local maximum and local minimum values of $f(x) = 3x^4 + 4x^3 - 12x^2 + 12$.

Solution:

$$f'(x) = 12x^3 + 12x^2 - 24x = 12x(x^2 + x - 2) = 12x(x+2)(x-1)$$

Critical points: $x = 0, -2, 1$. $f''(x) = 36x^2 + 24x - 24$.
At $x = 0$: $f''(0) = -24 < 0 \Rightarrow$ local max, $f(0) = 12$.
At $x = -2$: $f''(-2) = 144 - 48 - 24 = 72 > 0 \Rightarrow$ local min, $f(-2) = 48 - 32 - 48 + 12 = -20$.
At $x = 1$: $f''(1) = 36 + 24 - 24 = 36 > 0 \Rightarrow$ local min, $f(1) = 3 + 4 - 12 + 12 = 7$.

$$\boxed{\text{Local max} = 12 \text{ at } x = 0; \quad \text{Local minima} = -20 \text{ at } x = -2 \text{ and } 7 \text{ at } x = 1}$$

Question 5: A tank with a rectangular base and rectangular sides, open at the top, is to be constructed so that its depth is 2 m and volume is 8 m^3. If building of the tank costs Rs. 70 per sq. m for the base and Rs. 45 per sq. m for the sides, find the cost of the least expensive tank.

Solution: Depth $= 2$, volume $= 8 \Rightarrow$ base area $\times 2 = 8 \Rightarrow$ base area $= 4$. Let base sides be x, y with $xy = 4 \Rightarrow y = \dfrac{4}{x}$.
Base cost $= 70 \times xy = 70 \times 4 = 280$ (this is *constant*, since $xy = 4$ always).
Side area $= 2(\text{depth})(x + y) = 4(x + y)$. Side cost $= 45 \times 4(x + y) = 180\left(x + \dfrac{4}{x}\right)$.

Total cost: $C(x) = 180\left(x + \dfrac{4}{x}\right) + 280$.

$$C'(x) = 180\left(1 - \frac{4}{x^2}\right) = 0 \;\Rightarrow\; x^2 = 4 \;\Rightarrow\; x = 2, \quad y = \frac{4}{2} = 2$$

$C''(x) = \dfrac{1440}{x^3} > 0 \Rightarrow$ minimum. So the base is a 2 m $\times$ 2 m square.

$$C(2) = 180(2 + 2) + 280 = 720 + 280 = \boxed{\text{Rs. } 1000}$$

Question 6: A manufacturer can sell x items at a price of Rs. $\left(5 - \dfrac{x}{100}\right)$ each. The cost price of x items is Rs. $\left(\dfrac{x}{5} + 500\right)$. Find the number of items he should sell to earn maximum profit.

Solution: Revenue $R(x) = x\left(5 - \frac{x}{100}\right) = 5x - \frac{x^2}{100}$. Cost $C(x) = \frac{x}{5} + 500$.

$$P(x) = R(x) - C(x) = 5x - \frac{x^2}{100} - \frac{x}{5} - 500 = \frac{24x}{5} - \frac{x^2}{100} - 500$$

$$P'(x) = \frac{24}{5} - \frac{x}{50} = 0 \Rightarrow x = 240$$

$$P''(x) = -\frac{1}{50} < 0 \Rightarrow \text{maximum}$$

$$\boxed{\text{Selling 240 items gives maximum profit}}$$

Question 7: Show that the height of the cylinder of maximum volume that can be inscribed in a sphere of radius R is $\frac{2R}{\sqrt{3}}$, and find the maximum volume.

Solution: Let the cylinder have height h, radius r, inscribed in the sphere: $r^2 + \left(\frac{h}{2}\right)^2 = R^2 \Rightarrow r^2 = R^2 - \frac{h^2}{4}$.

$$V = \pi r^2 h = \pi h\left(R^2 - \frac{h^2}{4}\right) = \pi R^2 h - \frac{\pi h^3}{4}$$

$$\frac{dV}{dh} = \pi R^2 - \frac{3\pi h^2}{4} = 0 \Rightarrow h^2 = \frac{4R^2}{3} \Rightarrow h = \frac{2R}{\sqrt{3}}$$

$\frac{d^2V}{dh^2} = -\frac{3\pi h}{2} < 0 \Rightarrow$ maximum.

$$V_{max} = \pi R^2\left(\frac{2R}{\sqrt{3}}\right) - \frac{\pi}{4}\left(\frac{2R}{\sqrt{3}}\right)^3 = \frac{2\pi R^3}{\sqrt{3}} - \frac{2\pi R^3}{3\sqrt{3}} = \frac{4\pi R^3}{3\sqrt{3}} = \boxed{\frac{4\pi R^3}{3\sqrt{3}}}$$

Hence proved. ■

Question 8: The maximum value of $[x(x-1)+1]^{1/3}$, $0 \le x \le 1$ is:
(A) $\left(\frac{1}{3}\right)^{1/3}$ (B) $\frac{1}{2}$ (C) 1 (D) 0

Solution: Let $g(x) = x(x-1)+1 = x^2 - x + 1$. $g'(x) = 2x - 1 = 0 \Rightarrow x = \frac{1}{2}$ (this is a minimum since $g'' = 2 > 0$).

Checking endpoints: $g(0) = 1$, $g(1) = 1$. So the maximum value of $g(x)$ on $[0, 1]$ is 1 (at both endpoints).

Maximum of $g(x)^{1/3} = 1^{1/3} = 1$.

$$\boxed{\text{Option (C) is correct}}$$

Question 9: It is given that at $x = 1$, the function $f(x) = x^4 - 62x^2 + ax + 9$ attains its maximum value on the interval $[0, 2]$. Find the value of a.

Solution: $f'(x) = 4x^3 - 124x + a$. Since f has a maximum at $x = 1$ (an interior point), $f'(1) = 0$:

$$4(1) - 124(1) + a = 0 \;\Rightarrow\; a = 124 - 4 = \boxed{120}$$

Question 10: The function $f(x) = x^x$ has a stationary point at:

(A) $x = e$ (B) $x = \dfrac{1}{e}$ (C) $x = 1$ (D) $x = \sqrt{e}$

Solution: Let $y = x^x \Rightarrow \ln y = x \ln x$. Differentiating: $\dfrac{1}{y}\dfrac{dy}{dx} = \ln x + 1$, so $\dfrac{dy}{dx} = x^x(\ln x + 1)$.

Stationary point: $\ln x + 1 = 0 \Rightarrow x = \dfrac{1}{e}$.

$$\boxed{\text{Option (B) is correct}}$$

Question 11: The maximum value of $\left(\dfrac{1}{x}\right)^x$ is:

(A) e (B) $e^{1/e}$ (C) e^e (D) $\left(\dfrac{1}{e}\right)^{1/e}$

Solution: Let $f(x) = x^{-x} \Rightarrow \ln f = -x \ln x$. Differentiating: $\dfrac{f'}{f} = -(\ln x + 1) = 0 \Rightarrow x = \dfrac{1}{e}$.

$$f\left(\frac{1}{e}\right) = \left(\frac{1}{1/e}\right)^{1/e} = e^{1/e}$$

$$\boxed{\text{Option (B) is correct}}$$

Question 12: Find the maximum area of an isosceles triangle inscribed in the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ with its vertex at one end of the major axis.

Solution: Place the vertex at $(a, 0)$. Let the other two (symmetric) vertices be $(a\cos\theta,\ \pm b\sin\theta)$ on the ellipse.

Base of the triangle $= 2b\sin\theta$; height (horizontal distance) $= a - a\cos\theta = a(1-\cos\theta)$.

$$A(\theta) = \frac{1}{2} \times 2b\sin\theta \times a(1-\cos\theta) = ab\sin\theta(1-\cos\theta)$$

$$A'(\theta) = ab\big[\cos\theta(1-\cos\theta) + \sin\theta\cdot\sin\theta\big] = ab\big[\cos\theta - \cos^2\theta + \sin^2\theta\big] = ab\big[\cos\theta - \cos^2\theta + 1 - \cos^2\theta\big]$$

$$= ab\big[1 + \cos\theta - 2\cos^2\theta\big] = ab(1-\cos\theta)(1+2\cos\theta)$$

Setting $A'(\theta) = 0$ (excluding $\cos\theta = 1$, which gives zero area): $1+2\cos\theta = 0 \Rightarrow \cos\theta = -\dfrac{1}{2} \Rightarrow \theta = \dfrac{2\pi}{3}$.

Then $\sin\theta = \dfrac{\sqrt{3}}{2}$, and:

$$A_{max} = ab\cdot\frac{\sqrt{3}}{2}\left(1+\frac{1}{2}\right) = ab\cdot\frac{\sqrt{3}}{2}\cdot\frac{3}{2} = \boxed{\frac{3\sqrt{3}}{4}ab}$$

Quick Revision: Formula Sheet

1. Rate of change

$$\frac{dV}{dt} = \frac{dV}{dr} \cdot \frac{dr}{dt} \quad \text{(chain rule for related rates)}$$

2. Increasing/decreasing test

$$f'(x) \geq 0 \Rightarrow \text{increasing}, \qquad f'(x) \leq 0 \Rightarrow \text{decreasing on that interval}$$

3. Tangent & normal at (x_0, y_0), slope $m = f'(x_0)$

$$\text{Tangent: } y - y_0 = m(x - x_0) \qquad \text{Normal: } y - y_0 = -\frac{1}{m}(x - x_0)$$

4. Approximation

$$f(x + \Delta x) \approx f(x) + f'(x)\Delta x$$

5. Second derivative test at critical point c

$$f''(c) < 0 \Rightarrow \text{local max}, \qquad f''(c) > 0 \Rightarrow \text{local min}, \qquad f''(c) = 0 \Rightarrow \text{test fails}$$

6. Useful mensuration formulas for optimization problems

Solid	**Volume**	**Surface Area**
Sphere (radius r)	$\frac{4}{3}\pi r^3$	$4\pi r^2$
Cylinder (r, h)	$\pi r^2 h$	$2\pi rh + 2\pi r^2$ (total)
Cone (r, h, l)	$\frac{1}{3}\pi r^2 h$	$\pi rl + \pi r^2$ (total), $l = \sqrt{r^2 + h^2}$
Cube (side x)	x^3	$6x^2$
Cuboid (l, b, h)	lbh	$2(lb + bh + hl)$

Common mistakes to avoid

- In optimization problems, always reduce to *one* variable using the given constraint before differentiating — differentiating a two-variable expression directly is a common error.
- Don't forget to *verify* whether a critical point gives a maximum or minimum (via first/second derivative test) — a critical point alone doesn't guarantee an extremum.
- For absolute max/min on a closed interval $[a, b]$, always check the *endpoints* too, not just interior critical points.
- When a curve is defined parametrically, use $\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta}$ — don't try to eliminate the parameter unless it's easy.

— End of Chapter 6: Application of Derivatives —
Complete NCERT Solutions • Exercises 6.1 to 6.5 & Miscellaneous Exercise