NCERT CLASS 12 MATHEMATICS

Chapter-wise Solutions Series

CHAPTER 4

# DETERMINANTS

Complete Step-by-Step NCERT Solutions

### What's Inside

- Concept summary before every exercise
- Complete, verified solutions – Exercises 4.1 to 4.6
- Miscellaneous Exercise fully solved
- Clean step-by-step working for every problem
- Quick formula sheet for last-minute revision

Prepared for Class 12 CBSE / NCERT Mathematics students

Suitable for board exam preparation and quick revision

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**How to use this booklet:** Each exercise opens with a short concept box recapping the formulas you need. Every question is shown in a blue box, immediately followed by its complete worked solution in a green box. Work through the problems yourself first, then check your steps against the solution.

## Chapter Overview

A **determinant** is a unique number (scalar) associated with every square matrix. If $A$ is a square matrix, its determinant is denoted $|A|$ or $\det(A)$. Determinants are used to check whether a matrix is invertible, to solve systems of linear equations, and to compute areas of triangles/regions in coordinate geometry.

### Key Concepts & Formulas

1. **Determinant of order 2:** For $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$,

$$|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

2. **Determinant of order 3** (expansion along Row 1):

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

3. **Minor** $M_{ij}$: determinant obtained by deleting row $i$ and column $j$.
4. **Cofactor** $C_{ij} = (-1)^{i+j}M_{ij}$.
5. **Area of a triangle** with vertices $(x_1, y_1), (x_2, y_2), (x_3, y_3)$:

$$\Delta = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

6. **Adjoint of** $A$**:** transpose of the matrix of cofactors of $A$.
7. **Inverse of** $A$**:** $A^{-1} = \dfrac{1}{|A|}\,\mathrm{adj}(A)$, defined only when $|A| \neq 0$ (such a matrix is called *non-singular*).
8. **Solving** $AX = B$ **by matrix method:** $X = A^{-1}B$, provided $|A| \neq 0$ (unique solution).

# Exercise 4.1 | Determinants of Order 2 and 3

## Key Concepts & Formulas

Evaluate $2 \times 2$ determinants using $|A| = a_{11}a_{22} - a_{12}a_{21}$, and $3 \times 3$ determinants by expanding along any row or column (choose the one with the most zeros to save work).

**Question 1.** Evaluate the determinant $\begin{vmatrix} 2 & 4 \\ -5 & -1 \end{vmatrix}$.

**Solution:** Using $|A| = a_{11}a_{22} - a_{12}a_{21}$:

$$\begin{vmatrix} 2 & 4 \\ -5 & -1 \end{vmatrix} = (2)(-1) - (4)(-5) = -2 + 20 = \boxed{18}$$

**Question 2.** Evaluate the determinants:
(i) $\begin{vmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{vmatrix}$ (ii) $\begin{vmatrix} x^2 - x + 1 & x - 1 \\ x + 1 & x + 1 \end{vmatrix}$

**Solution:** (i) $\begin{vmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{vmatrix} = \cos\theta \cdot \cos\theta - (-\sin\theta)(\sin\theta) = \cos^2\theta + \sin^2\theta = \boxed{1}$

(ii) $\begin{vmatrix} x^2 - x + 1 & x - 1 \\ x + 1 & x + 1 \end{vmatrix} = (x^2 - x + 1)(x + 1) - (x - 1)(x + 1)$

$$= (x^3 + 1) - (x^2 - 1) \qquad [\text{since } (x^2 - x + 1)(x + 1) = x^3 + 1]$$
$$= \boxed{x^3 - x^2 + 2}$$

**Question 3.** If $\begin{vmatrix} x & 2 \\ 18 & x \end{vmatrix} = \begin{vmatrix} 6 & 2 \\ 18 & 6 \end{vmatrix}$, find the value(s) of $x$.

**Solution:** Evaluating both determinants:

$$x^2 - 36 = 36 - 36 \implies x^2 - 36 = 0 \implies x^2 = 36$$

$$\boxed{x = \pm 6}$$

**Question 4.** If $\begin{vmatrix} 2 & 4 \\ 5 & 1 \end{vmatrix} = \begin{vmatrix} 2x & 4 \\ 6 & x \end{vmatrix}$, find the value(s) of $x$.

**Solution:** LHS $= 2(1) - 4(5) = 2 - 20 = -18$. RHS $= 2x(x) - 4(6) = 2x^2 - 24$.

$$-18 = 2x^2 - 24 \implies 2x^2 = 6 \implies x^2 = 3$$

$$\boxed{x = \pm\sqrt{3}}$$

**Question 5.** If $A = \begin{pmatrix} 1 & 1 & -2 \\ 2 & 1 & -3 \\ 5 & 4 & -9 \end{pmatrix}$, find $|A|$.

**Solution:** Expanding along Row 1:

$$\begin{aligned} |A| &= 1\begin{vmatrix} 1 & -3 \\ 4 & -9 \end{vmatrix} - 1\begin{vmatrix} 2 & -3 \\ 5 & -9 \end{vmatrix} + (-2)\begin{vmatrix} 2 & 1 \\ 5 & 4 \end{vmatrix} \\ &= 1(-9+12) - 1(-18+15) - 2(8-5) \\ &= 1(3) - 1(-3) - 2(3) \\ &= 3 + 3 - 6 \\ &= \boxed{0} \end{aligned}$$

**Question 6.** Evaluate the following determinants:

(i) $\begin{vmatrix} 3 & -1 & -2 \\ 0 & 0 & -1 \\ 3 & -5 & 0 \end{vmatrix}$ (ii) $\begin{vmatrix} 3 & -4 & 5 \\ 1 & 1 & -2 \\ 2 & 3 & 1 \end{vmatrix}$

(iii) $\begin{vmatrix} 0 & 1 & 2 \\ -1 & 0 & -3 \\ -2 & 3 & 0 \end{vmatrix}$ (iv) $\begin{vmatrix} 2 & -1 & -2 \\ 0 & 2 & -1 \\ 3 & -5 & 0 \end{vmatrix}$

**Solution: (i)** Expanding along Row 1:

$$\begin{aligned} \Delta &= 3\begin{vmatrix} 0 & -1 \\ -5 & 0 \end{vmatrix} - (-1)\begin{vmatrix} 0 & -1 \\ 3 & 0 \end{vmatrix} + (-2)\begin{vmatrix} 0 & 0 \\ 3 & -5 \end{vmatrix} \\ &= 3(0-5) + 1(0+3) - 2(0-0) = 3(-5) + 3 - 0 = -15 + 3 = \boxed{-12} \end{aligned}$$

**(ii)** Expanding along Row 1:

$$\begin{aligned} \Delta &= 3\begin{vmatrix} 1 & -2 \\ 3 & 1 \end{vmatrix} - (-4)\begin{vmatrix} 1 & -2 \\ 2 & 1 \end{vmatrix} + 5\begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix} \\ &= 3(1+6) + 4(1+4) + 5(3-2) = 3(7) + 4(5) + 5(1) = 21 + 20 + 5 = \boxed{46} \end{aligned}$$

**(iii)** Expanding along Row 1:

$$\begin{aligned} \Delta &= 0\begin{vmatrix} 0 & -3 \\ 3 & 0 \end{vmatrix} - 1\begin{vmatrix} -1 & -3 \\ -2 & 0 \end{vmatrix} + 2\begin{vmatrix} -1 & 0 \\ -2 & 3 \end{vmatrix} \\ &= 0 - 1(0-6) + 2(-3-0) = 0 + 6 - 6 = \boxed{0} \end{aligned}$$

**(iv)** Expanding along Row 1:

$$\begin{aligned} \Delta &= 2\begin{vmatrix} 2 & -1 \\ -5 & 0 \end{vmatrix} - (-1)\begin{vmatrix} 0 & -1 \\ 3 & 0 \end{vmatrix} + (-2)\begin{vmatrix} 0 & 2 \\ 3 & -5 \end{vmatrix} \\ &= 2(0-5) + 1(0+3) - 2(0-6) = -10 + 3 + 12 = \boxed{5} \end{aligned}$$

# Exercise 4.2 | Properties of Determinants

## Key Concepts & Formulas

Useful properties: (a) if any two rows/columns are identical the determinant is 0; (b) if a row/column is entirely zero the determinant is 0; (c) $R_i \to R_i + kR_j$ does not change the determinant's value; (d) taking a common factor out of a row/column multiplies the determinant by that factor. These let us simplify a determinant *before* expanding.

**Question 1.** Using properties of determinants, prove that $\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = (a-b)(b-c)(c-a)$.

**Solution:** Apply $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$:

$$\Delta = \begin{vmatrix} 1 & a & a^2 \\ 0 & b-a & b^2-a^2 \\ 0 & c-a & c^2-a^2 \end{vmatrix}$$

Expanding along Column 1:

$$\begin{aligned} \Delta &= 1 \cdot \begin{vmatrix} b-a & b^2-a^2 \\ c-a & c^2-a^2 \end{vmatrix} \\ &= (b-a)(c^2-a^2) - (b^2-a^2)(c-a) \\ &= (b-a)(c-a)(c+a) - (b-a)(b+a)(c-a) \\ &= (b-a)(c-a)\big[(c+a)-(b+a)\big] \\ &= (b-a)(c-a)(c-b) \\ &= \boxed{(a-b)(b-c)(c-a)} \end{aligned}$$

**Question 2.** Using properties of determinants, prove that $\begin{vmatrix} x+4 & 2x & 2x \\ 2x & x+4 & 2x \\ 2x & 2x & x+4 \end{vmatrix} = (5x+4)(4-x)^2$.

**Solution:** Apply $C_1 \to C_1 + C_2 + C_3$. Each row now sums to $x+4+2x+2x = 5x+4$:

$$\Delta = (5x+4)\begin{vmatrix} 1 & 2x & 2x \\ 1 & x+4 & 2x \\ 1 & 2x & x+4 \end{vmatrix}$$

Now apply $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$:

$$\Delta = (5x+4)\begin{vmatrix} 1 & 2x & 2x \\ 0 & 4-x & 0 \\ 0 & 0 & 4-x \end{vmatrix}$$

Expanding along Column 1:

$$\Delta = (5x+4) \cdot 1 \cdot \big[(4-x)(4-x) - 0\big] = \boxed{(5x+4)(4-x)^2}$$

**Question 3.** Using properties of determinants, prove that $\begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix} = (a+b+c)^3$.

**Solution:** Apply $R_1 \to R_1 + R_2 + R_3$. Every entry of the new Row 1 becomes $a+b+c$ (e.g. Column 1: $(a-b-c)+2b+2c = a+b+c$):

$$\Delta = (a+b+c)\begin{vmatrix} 1 & 1 & 1 \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$$

Apply $C_2 \to C_2 - C_1$ and $C_3 \to C_3 - C_1$:

$$\Delta = (a+b+c)\begin{vmatrix} 1 & 0 & 0 \\ 2b & -(a+b+c) & 0 \\ 2c & 0 & -(a+b+c) \end{vmatrix}$$

Expanding along Row 1:

$$\Delta = (a+b+c)\cdot 1 \cdot \left[(a+b+c)^2 - 0\right] = \boxed{(a+b+c)^3}$$

**Question 4.** Using properties of determinants, prove that $\begin{vmatrix} a-b & b-c & c-a \\ b-c & c-a & a-b \\ c-a & a-b & b-c \end{vmatrix} = 0$.

**Solution:** Apply $C_1 \to C_1 + C_2 + C_3$. Each row sum is $(a-b)+(b-c)+(c-a) = 0$, so every row of the new Column 1 becomes 0:

$$\Delta = \begin{vmatrix} 0 & b-c & c-a \\ 0 & c-a & a-b \\ 0 & a-b & b-c \end{vmatrix}$$

Since Column 1 is entirely zero, $\Delta = \boxed{0}$.

**Question 5.** Using properties of determinants, prove that $\begin{vmatrix} 3a & -a+b & -a+c \\ -b+a & 3b & -b+c \\ -c+a & -c+b & 3c \end{vmatrix} = 3(a+b+c)(ab+bc+ca)$.

**Solution:** Apply $C_1 \to C_1 + C_2 + C_3$. Each row now sums to $a+b+c$ (e.g. Row 1: $3a + (-a+b) + (-a+c) = a+b+c$):

$$\Delta = (a+b+c)\begin{vmatrix} 1 & -a+b & -a+c \\ 1 & 3b & -b+c \\ 1 & -c+b & 3c \end{vmatrix}$$

Apply $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$:

$$\Delta = (a+b+c)\begin{vmatrix} 1 & -a+b & -a+c \\ 0 & a+2b & a-b \\ 0 & a-c & a+2c \end{vmatrix}$$

Expanding along Column 1:

$$\Delta = (a+b+c)\Big[(a+2b)(a+2c) - (a-b)(a-c)\Big]$$

Now $(a+2b)(a+2c) = a^2+2ab+2ac+4bc$ and $(a-b)(a-c) = a^2-ab-ac+bc$, so their difference is $3ab+3ac+3bc = 3(ab+bc+ca)$. Hence

$$\Delta = \boxed{3(a+b+c)(ab+bc+ca)}$$

# Exercise 4.3 | Area of a Triangle

## Key Concepts & Formulas

Area of a triangle with vertices $(x_1, y_1), (x_2, y_2), (x_3, y_3)$:

$$\Delta = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

Since area is always positive, take the absolute value of the determinant. Three points are **collinear** if this determinant equals 0.

**Question 1.** Find the area of the triangle with vertices $(1, 0)$, $(6, 0)$, $(4, 3)$.

**Solution:** Using the formula $\Delta = \frac{1}{2}|x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)|$ with $(x_1, y_1) = (1, 0), (x_2, y_2) = (6, 0), (x_3, y_3) = (4, 3)$:

$$\begin{aligned} \text{Area} &= \frac{1}{2}|1(0 - 3) + 6(3 - 0) + 4(0 - 0)| \\ &= \frac{1}{2}| - 3 + 18 + 0| = \frac{1}{2}(15) = \boxed{7.5 \text{ sq units}} \end{aligned}$$

**Question 2.** Find the area of the triangle with vertices $(2, 7)$, $(1, 1)$, $(10, 8)$.

**Solution:**

$$\begin{aligned} \text{Area} &= \frac{1}{2}|x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)| \\ &= \frac{1}{2}|2(1 - 8) + 1(8 - 7) + 10(7 - 1)| \\ &= \frac{1}{2}|2(-7) + 1(1) + 10(6)| \\ &= \frac{1}{2}| - 14 + 1 + 60| = \frac{1}{2}(47) = \boxed{23.5 \text{ sq units}} \end{aligned}$$

**Question 3.** Find the area of the triangle with vertices $(-2, -3)$, $(3, 2)$, $(-1, -8)$.

**Solution:**

$$\begin{aligned} \text{Area} &= \frac{1}{2}|(-2)(2 - (-8)) + 3((-8) - (-3)) + (-1)((-3) - 2)| \\ &= \frac{1}{2}|(-2)(10) + 3(-5) + (-1)(-5)| \\ &= \frac{1}{2}| - 20 - 15 + 5| = \frac{1}{2}| - 30| = \boxed{15 \text{ sq units}} \end{aligned}$$

**Question 4.** Using determinants, find the equation of the line joining the points $(3, 1)$ and $(9, 3)$.

**Solution:** Let $(x, y)$ be any point on the line. Then $(x, y), (3, 1), (9, 3)$ are collinear, so

$$\begin{vmatrix} x & y & 1 \\ 3 & 1 & 1 \\ 9 & 3 & 1 \end{vmatrix} = 0$$

Expanding along Row 1:

$$x(1 - 3) - y(3 - 9) + 1(9 - 9) = 0$$

$$-2x + 6y + 0 = 0$$

$$\boxed{x - 3y = 0}$$

*Check:* both $(3, 1)$ and $(9, 3)$ satisfy $x - 3y = 0$. ✓

**Question 5.** Find the value of $k$ if the area of the triangle with vertices $(k, 0)$, $(4, 0)$, $(0, 2)$ is 4 sq units.

**Solution:**

$$\text{Area} = \frac{1}{2}|k(0 - 2) + 4(2 - 0) + 0(0 - 0)| = \frac{1}{2}| - 2k + 8|$$

Given Area $= 4$:

$$\frac{1}{2}|-2k + 8| = 4 \implies |-2k + 8| = 8$$

This gives two cases:

$$-2k + 8 = 8 \ \Rightarrow\ k = 0 \qquad \text{or} \qquad -2k + 8 = -8 \ \Rightarrow\ k = 8$$

$$\boxed{k = 0 \text{ or } k = 8}$$

## Exercise 4.4 | Minors and Cofactors

### Key Concepts & Formulas

**Minor** $M_{ij}$ of element $a_{ij}$: the determinant left after deleting row $i$ and column $j$. **Cofactor** $C_{ij} = (-1)^{i+j} M_{ij}$. A determinant can be evaluated by summing (element × its cofactor) along *any* row or column.

**Question 1.** Write the minors and cofactors of the elements of the determinant $\begin{vmatrix} 2 & -4 \\ 0 & 3 \end{vmatrix}$.

**Solution:**

$$\begin{aligned} M_{11} &= 3, \quad C_{11} = (+1)(3) = 3 \\ M_{12} &= 0, \quad C_{12} = (-1)(0) = 0 \\ M_{21} &= -4, \quad C_{21} = (-1)(-4) = 4 \\ M_{22} &= 2, \quad C_{22} = (+1)(2) = 2 \end{aligned}$$

**Question 2.** Write the minors and cofactors of the elements of the determinant $\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$.

**Solution:**

$$\begin{aligned}
M_{11} &= \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1, & C_{11} &= 1 \\
M_{12} &= \begin{vmatrix} 0 & 0 \\ 0 & 1 \end{vmatrix} = 0, & C_{12} &= 0 \\
M_{13} &= \begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix} = 0, & C_{13} &= 0 \\
M_{21} &= \begin{vmatrix} 0 & 0 \\ 0 & 1 \end{vmatrix} = 0, & C_{21} &= 0 \\
M_{22} &= \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1, & C_{22} &= 1 \\
M_{23} &= \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} = 0, & C_{23} &= 0 \\
M_{31} &= \begin{vmatrix} 0 & 0 \\ 1 & 0 \end{vmatrix} = 0, & C_{31} &= 0 \\
M_{32} &= \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} = 0, & C_{32} &= 0 \\
M_{33} &= \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1, & C_{33} &= 1
\end{aligned}$$

**Question 3.** Using cofactors of elements of the second row, evaluate $\Delta = \begin{vmatrix} 5 & 3 & 8 \\ 2 & 0 & 1 \\ 1 & 2 & 3 \end{vmatrix}$.

**Solution:** Row 2 elements: $a_{21} = 2,\ a_{22} = 0,\ a_{23} = 1$.

$$C_{21} = (-1)^{2+1}\begin{vmatrix} 3 & 8 \\ 2 & 3 \end{vmatrix} = -(9 - 16) = 7$$
$$C_{22} = (-1)^{2+2}\begin{vmatrix} 5 & 8 \\ 1 & 3 \end{vmatrix} = (15 - 8) = 7$$
$$C_{23} = (-1)^{2+3}\begin{vmatrix} 5 & 3 \\ 1 & 2 \end{vmatrix} = -(10 - 3) = -7$$

$$\Delta = a_{21}C_{21} + a_{22}C_{22} + a_{23}C_{23} = 2(7) + 0(7) + 1(-7) = 14 + 0 - 7 = \boxed{7}$$

**Question 4.** Using cofactors of elements of the third column, evaluate $\Delta = \begin{vmatrix} 1 & x & yz \\ 1 & y & zx \\ 1 & z & xy \end{vmatrix}$.

**Solution:** Column 3 elements: $a_{13} = yz,\ a_{23} = zx,\ a_{33} = xy$.

$$C_{13} = (+1)\begin{vmatrix} 1 & y \\ 1 & z \end{vmatrix} = z - y$$
$$C_{23} = (-1)\begin{vmatrix} 1 & x \\ 1 & z \end{vmatrix} = -(z - x) = x - z$$
$$C_{33} = (+1)\begin{vmatrix} 1 & x \\ 1 & y \end{vmatrix} = y - x$$

$$\begin{aligned}\Delta &= yz(z - y) + zx(x - z) + xy(y - x) \\ &= yz^2 - y^2z + zx^2 - xz^2 + xy^2 - x^2y\end{aligned}$$

Grouping and factoring, this expression simplifies to

$$\Delta = \boxed{(x - y)(y - z)(z - x)}$$

# Exercise 4.5 | Adjoint and Inverse of a Matrix

## Key Concepts & Formulas

**Adjoint:** $\text{adj}(A)$ = transpose of the cofactor matrix of $A$. **Property:** $A(\text{adj } A) = (\text{adj } A)A = |A|\, I$. **Inverse:** $A^{-1} = \frac{1}{|A|}\text{adj}(A)$, valid only when $|A| \neq 0$.

**Question 1.** Find the adjoint of the matrix $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$.

**Solution:** For a $2 \times 2$ matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, the adjoint is $\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$. So

$$\text{adj}(A) = \boxed{\begin{pmatrix} 4 & -2 \\ -3 & 1 \end{pmatrix}}$$

**Question 2.** Find the adjoint of the matrix $A = \begin{pmatrix} 1 & -1 & 2 \\ 2 & 3 & 5 \\ -2 & 0 & 1 \end{pmatrix}$.

**Solution:** Compute all nine cofactors:

$$C_{11} = \begin{vmatrix} 3 & 5 \\ 0 & 1 \end{vmatrix} = 3, \qquad C_{12} = -\begin{vmatrix} 2 & 5 \\ -2 & 1 \end{vmatrix} = -12, \qquad C_{13} = \begin{vmatrix} 2 & 3 \\ -2 & 0 \end{vmatrix} = 6$$

$$C_{21} = -\begin{vmatrix} -1 & 2 \\ 0 & 1 \end{vmatrix} = 1, \qquad C_{22} = \begin{vmatrix} 1 & 2 \\ -2 & 1 \end{vmatrix} = 5, \qquad C_{23} = -\begin{vmatrix} 1 & -1 \\ -2 & 0 \end{vmatrix} = 2$$

$$C_{31} = \begin{vmatrix} -1 & 2 \\ 3 & 5 \end{vmatrix} = -11, \qquad C_{32} = -\begin{vmatrix} 1 & 2 \\ 2 & 5 \end{vmatrix} = -1, \qquad C_{33} = \begin{vmatrix} 1 & -1 \\ 2 & 3 \end{vmatrix} = 5$$

Cofactor matrix $= \begin{pmatrix} 3 & -12 & 6 \\ 1 & 5 & 2 \\ -11 & -1 & 5 \end{pmatrix}$. The adjoint is its transpose:

$$\text{adj}(A) = \boxed{\begin{pmatrix} 3 & 1 & -11 \\ -12 & 5 & -1 \\ 6 & 2 & 5 \end{pmatrix}}$$

**Question 3.** Verify $A(\text{adj } A) = (\text{adj } A)A = |A|\, I$ for $A = \begin{pmatrix} 2 & 3 \\ -4 & -6 \end{pmatrix}$.

**Solution:**

$$|A| = 2(-6) - 3(-4) = -12 + 12 = 0, \qquad \text{adj}(A) = \begin{pmatrix} -6 & -3 \\ 4 & 2 \end{pmatrix}$$

$$A \cdot \text{adj}(A) = \begin{pmatrix} 2 & 3 \\ -4 & -6 \end{pmatrix}\begin{pmatrix} -6 & -3 \\ 4 & 2 \end{pmatrix} = \begin{pmatrix} -12+12 & -6+6 \\ 24-24 & 12-12 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\text{adj}(A) \cdot A = \begin{pmatrix} -6 & -3 \\ 4 & 2 \end{pmatrix}\begin{pmatrix} 2 & 3 \\ -4 & -6 \end{pmatrix} = \begin{pmatrix} -12+12 & -18+18 \\ 8-8 & 12-12 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Since $|A|I = 0 \cdot I = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, we have $A(\text{adj } A) = (\text{adj } A)A = |A|I$. $\boxed{\text{Verified}}$

**Question 4.** Find the inverse of $A = \begin{pmatrix} 2 & -2 \\ 4 & 3 \end{pmatrix}$.

**Solution:**

$$|A| = 2(3) - (-2)(4) = 6 + 8 = 14 \neq 0 \quad (\text{so } A^{-1} \text{ exists})$$

$$\text{adj}(A) = \begin{pmatrix} 3 & 2 \\ -4 & 2 \end{pmatrix}$$

$$A^{-1} = \frac{1}{|A|}\text{adj}(A) = \boxed{\frac{1}{14}\begin{pmatrix} 3 & 2 \\ -4 & 2 \end{pmatrix}}$$

**Question 5.** Find the inverse of $A = \begin{pmatrix} 1 & 0 & 0 \\ 3 & 3 & 0 \\ 5 & 2 & -1 \end{pmatrix}$.

**Solution:** Expanding along Row 1: $|A| = 1(3(-1) - 0(2)) = 1(-3) = -3 \neq 0$.
Cofactors:

$$C_{11} = \begin{vmatrix} 3 & 0 \\ 2 & -1 \end{vmatrix} = -3, \qquad C_{12} = -\begin{vmatrix} 3 & 0 \\ 5 & -1 \end{vmatrix} = 3, \qquad C_{13} = \begin{vmatrix} 3 & 3 \\ 5 & 2 \end{vmatrix} = -9$$

$$C_{21} = -\begin{vmatrix} 0 & 0 \\ 2 & -1 \end{vmatrix} = 0, \qquad C_{22} = \begin{vmatrix} 1 & 0 \\ 5 & -1 \end{vmatrix} = -1, \qquad C_{23} = -\begin{vmatrix} 1 & 0 \\ 5 & 2 \end{vmatrix} = -2$$

$$C_{31} = \begin{vmatrix} 0 & 0 \\ 3 & 0 \end{vmatrix} = 0, \qquad C_{32} = -\begin{vmatrix} 1 & 0 \\ 3 & 0 \end{vmatrix} = 0, \qquad C_{33} = \begin{vmatrix} 1 & 0 \\ 3 & 3 \end{vmatrix} = 3$$

$$\text{adj}(A) = \begin{pmatrix} -3 & 0 & 0 \\ 3 & -1 & 0 \\ -9 & -2 & 3 \end{pmatrix}$$

$$A^{-1} = \frac{1}{-3}\begin{pmatrix} -3 & 0 & 0 \\ 3 & -1 & 0 \\ -9 & -2 & 3 \end{pmatrix} = \boxed{\begin{pmatrix} 1 & 0 & 0 \\ -1 & \frac{1}{3} & 0 \\ 3 & \frac{2}{3} & -1 \end{pmatrix}}$$

**Question 6.** If $A = \begin{pmatrix} 3 & 7 \\ 2 & 5 \end{pmatrix}$ and $B = \begin{pmatrix} 6 & 8 \\ 7 & 9 \end{pmatrix}$, verify $(AB)^{-1} = B^{-1}A^{-1}$.

**Solution:**

$$|A| = 15 - 14 = 1, \quad A^{-1} = \begin{pmatrix} 5 & -7 \\ -2 & 3 \end{pmatrix}$$

$$|B| = 54 - 56 = -2, \quad B^{-1} = \frac{1}{-2}\begin{pmatrix} 9 & -8 \\ -7 & 6 \end{pmatrix} = \begin{pmatrix} -4.5 & 4 \\ 3.5 & -3 \end{pmatrix}$$

$$AB = \begin{pmatrix} 3 & 7 \\ 2 & 5 \end{pmatrix}\begin{pmatrix} 6 & 8 \\ 7 & 9 \end{pmatrix} = \begin{pmatrix} 67 & 87 \\ 47 & 61 \end{pmatrix}, \qquad |AB| = 67(61) - 87(47) = -2$$

$$(AB)^{-1} = \frac{1}{-2}\begin{pmatrix} 61 & -87 \\ -47 & 67 \end{pmatrix} = \begin{pmatrix} -30.5 & 43.5 \\ 23.5 & -33.5 \end{pmatrix}$$

Now compute $B^{-1}A^{-1}$:

$$B^{-1}A^{-1} = \begin{pmatrix} -4.5 & 4 \\ 3.5 & -3 \end{pmatrix}\begin{pmatrix} 5 & -7 \\ -2 & 3 \end{pmatrix} = \begin{pmatrix} -30.5 & 43.5 \\ 23.5 & -33.5 \end{pmatrix}$$

Since both results are identical, $(AB)^{-1} = B^{-1}A^{-1}$. $\boxed{\text{Verified}}$

**Question 7.** Find the inverse of $A = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$.

**Solution:** Expanding along Row 1:

$$|A| = 2(4-1) - 1(2-1) + 1(1-2) = 2(3) - 1(1) + 1(-1) = 6 - 1 - 1 = 4$$

Cofactors (matrix is symmetric, so cofactor matrix is also symmetric):

$$C_{11} = \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = 3, \qquad C_{12} = -\begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} = -1, \qquad C_{13} = \begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix} = -1$$

$$C_{21} = -1, \qquad C_{22} = 3, \qquad C_{23} = -1$$

$$C_{31} = -1, \qquad C_{32} = -1, \qquad C_{33} = 3$$

$$\text{adj}(A) = \begin{pmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{pmatrix} \quad \text{(symmetric, so equals its own transpose)}$$

$$A^{-1} = \frac{1}{4}\begin{pmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{pmatrix}$$

# Exercise 4.6 | Applications: Solving a System of Linear Equations

## Key Concepts & Formulas

Write the system as $AX = B$. If $|A| \neq 0$, the system is consistent with a unique solution given by $X = A^{-1}B = \frac{1}{|A|}\text{adj}(A) \cdot B$.

**Question 1.** Solve the system of equations by matrix method: $2x - y = 5,\ \ x + y = 4$.

**Solution:** Write $AX = B$ with $A = \begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix}$, $X = \begin{pmatrix} x \\ y \end{pmatrix}$, $B = \begin{pmatrix} 5 \\ 4 \end{pmatrix}$.

$$|A| = 2(1) - (-1)(1) = 3 \neq 0 \quad \text{(unique solution exists)}$$

$$\text{adj}(A) = \begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix} \implies A^{-1} = \frac{1}{3}\begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix}$$

$$X = A^{-1}B = \frac{1}{3}\begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 5 \\ 4 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 9 \\ 3 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

$$\boxed{x = 3,\ y = 1}$$

**Question 2.** Solve the system of equations by matrix method: $x + 2y = 2,\ \ 2x + 3y = 3$.

**Solution:** $A = \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}$, $B = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$.

$$|A| = 3 - 4 = -1 \neq 0$$

$$\text{adj}(A) = \begin{pmatrix} 3 & -2 \\ -2 & 1 \end{pmatrix} \implies A^{-1} = \frac{1}{-1}\begin{pmatrix} 3 & -2 \\ -2 & 1 \end{pmatrix} = \begin{pmatrix} -3 & 2 \\ 2 & -1 \end{pmatrix}$$

$$X = A^{-1}B = \begin{pmatrix} -3 & 2 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} -6+6 \\ 4-3 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\boxed{x = 0,\ y = 1}$$

**Question 3.** Solve the system of equations by matrix method: $x - y + z = 4,\ \ 2x + y - 3z = 0,\ \ x + y + z = 2$.

**Solution:** $A = \begin{pmatrix} 1 & -1 & 1 \\ 2 & 1 & -3 \\ 1 & 1 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 4 \\ 0 \\ 2 \end{pmatrix}$.

$|A| = 1(1+3) + 1(2+3) + 1(2-1) = 4 + 5 + 1 = 10 \neq 0$.

Cofactors: $C_{11} = 4, C_{12} = -5, C_{13} = 1, C_{21} = 2, C_{22} = 0, C_{23} = -2, C_{31} = 2, C_{32} = 5, C_{33} = 3$.

$$\text{adj}(A) = \begin{pmatrix} 4 & 2 & 2 \\ -5 & 0 & 5 \\ 1 & -2 & 3 \end{pmatrix} \implies A^{-1} = \frac{1}{10} \begin{pmatrix} 4 & 2 & 2 \\ -5 & 0 & 5 \\ 1 & -2 & 3 \end{pmatrix}$$

$$X = A^{-1}B = \frac{1}{10} \begin{pmatrix} 4(4) + 2(0) + 2(2) \\ -5(4) + 0(0) + 5(2) \\ 1(4) - 2(0) + 3(2) \end{pmatrix} = \frac{1}{10} \begin{pmatrix} 20 \\ -10 \\ 10 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$$

$$\boxed{x = 2,\ y = -1,\ z = 1}$$

*Check:* $2 - (-1) + 1 = 4\checkmark$, $2(2) + (-1) - 3(1) = 0\checkmark$, $2 + (-1) + 1 = 2\checkmark$

**Question 4.** Solve the system of equations by matrix method: $2x + 3y + 3z = 5$, $x - 2y + z = -4$, $3x - y - 2z = 3$.

**Solution:** $A = \begin{pmatrix} 2 & 3 & 3 \\ 1 & -2 & 1 \\ 3 & -1 & -2 \end{pmatrix}$, $B = \begin{pmatrix} 5 \\ -4 \\ 3 \end{pmatrix}$.
$|A| = 2(4 + 1) - 3(-2 - 3) + 3(-1 + 6) = 10 + 15 + 15 = 40 \neq 0$.
Cofactors: $C_{11} = 5, C_{12} = 5, C_{13} = 5, C_{21} = 3, C_{22} = -13, C_{23} = 11, C_{31} = 9, C_{32} = 1, C_{33} = -7$.

$$\text{adj}(A) = \begin{pmatrix} 5 & 3 & 9 \\ 5 & -13 & 1 \\ 5 & 11 & -7 \end{pmatrix} \implies A^{-1} = \frac{1}{40} \begin{pmatrix} 5 & 3 & 9 \\ 5 & -13 & 1 \\ 5 & 11 & -7 \end{pmatrix}$$

$$X = A^{-1}B = \frac{1}{40} \begin{pmatrix} 5(5) + 3(-4) + 9(3) \\ 5(5) - 13(-4) + 1(3) \\ 5(5) + 11(-4) - 7(3) \end{pmatrix} = \frac{1}{40} \begin{pmatrix} 40 \\ 80 \\ -40 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$$

$$\boxed{x = 1,\ y = 2,\ z = -1}$$

*Check:* $2(1) + 3(2) + 3(-1) = 5\checkmark$, $1 - 2(2) + (-1) = -4\checkmark$, $3(1) - 2 - 2(-1) = 3\checkmark$

**Question 5.** The cost of 4 kg onion, 3 kg wheat and 2 kg rice is Rs 60. The cost of 2 kg onion, 4 kg wheat and 6 kg rice is Rs 90. The cost of 6 kg onion, 2 kg wheat and 3 kg rice is Rs 70. Find the cost of each item per kg using matrices.

**Solution:** Let the cost per kg of onion, wheat, rice be $x, y, z$ respectively. Then:

$$4x + 3y + 2z = 60, \qquad 2x + 4y + 6z = 90, \qquad 6x + 2y + 3z = 70$$

$A = \begin{pmatrix} 4 & 3 & 2 \\ 2 & 4 & 6 \\ 6 & 2 & 3 \end{pmatrix}$, $B = \begin{pmatrix} 60 \\ 90 \\ 70 \end{pmatrix}$.
$|A| = 4(12 - 12) - 3(6 - 36) + 2(4 - 24) = 0 + 90 - 40 = 50 \neq 0$.
Cofactors: $C_{11} = 0, C_{12} = 30, C_{13} = -20, C_{21} = -5, C_{22} = 0, C_{23} = 10, C_{31} = 10, C_{32} = -20, C_{33} = 10$.

$$\text{adj}(A) = \begin{pmatrix} 0 & -5 & 10 \\ 30 & 0 & -20 \\ -20 & 10 & 10 \end{pmatrix} \implies A^{-1} = \frac{1}{50} \begin{pmatrix} 0 & -5 & 10 \\ 30 & 0 & -20 \\ -20 & 10 & 10 \end{pmatrix}$$

$$X = A^{-1}B = \frac{1}{50}\begin{pmatrix} 0(60) - 5(90) + 10(70) \\ 30(60) + 0(90) - 20(70) \\ -20(60) + 10(90) + 10(70) \end{pmatrix} = \frac{1}{50}\begin{pmatrix} 250 \\ 400 \\ 400 \end{pmatrix} = \begin{pmatrix} 5 \\ 8 \\ 8 \end{pmatrix}$$

$$\boxed{\text{Onion} = \text{Rs } 5/\text{kg}, \quad \text{Wheat} = \text{Rs } 8/\text{kg}, \quad \text{Rice} = \text{Rs } 8/\text{kg}}$$

# Miscellaneous Exercise

**Question 1.** If $A^{-1} = \begin{pmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{pmatrix}$, find $(AB)^{-1}$.

**Solution:** Use $(AB)^{-1} = B^{-1}A^{-1}$. First find $B^{-1}$.
$|B| = 1(3-0) - 2(-1-0) + (-2)(2-0) = 3 + 2 - 4 = 1$.
Cofactors of $B$: $C_{11} = 3, C_{12} = 1, C_{13} = 2, C_{21} = 2, C_{22} = 1, C_{23} = 2, C_{31} = 6, C_{32} = 2, C_{33} = 5$.

$$\text{adj}(B) = \begin{pmatrix} 3 & 2 & 6 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{pmatrix} \implies B^{-1} = \frac{1}{1}\begin{pmatrix} 3 & 2 & 6 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{pmatrix}$$

Now multiply:

$$(AB)^{-1} = B^{-1}A^{-1} = \begin{pmatrix} 3 & 2 & 6 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{pmatrix}\begin{pmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{pmatrix}$$

Row 1: $(9 - 30 + 30,\ -3 + 12 - 12,\ 3 - 10 + 12) = (9, -3, 5)$
Row 2: $(3 - 15 + 10,\ -1 + 6 - 4,\ 1 - 5 + 4) = (-2, 1, 0)$
Row 3: $(6 - 30 + 25,\ -2 + 12 - 10,\ 2 - 10 + 10) = (1, 0, 2)$

$$(AB)^{-1} = \boxed{\begin{pmatrix} 9 & -3 & 5 \\ -2 & 1 & 0 \\ 1 & 0 & 2 \end{pmatrix}}$$

**Question 2.** Solve the system of equations by matrix method: $x - y + 2z = 7,\ \ 3x + 4y - 5z = -5,\ \ 2x - y + 3z = 12$.

**Solution:** $A = \begin{pmatrix} 1 & -1 & 2 \\ 3 & 4 & -5 \\ 2 & -1 & 3 \end{pmatrix}$, $B = \begin{pmatrix} 7 \\ -5 \\ 12 \end{pmatrix}$.
$|A| = 1(12-5) + 1(9+10) + 2(-3-8) = 7 + 19 - 22 = 4 \neq 0$.
Cofactors: $C_{11} = 7, C_{12} = -19, C_{13} = -11, C_{21} = 1, C_{22} = -1, C_{23} = -1, C_{31} = -3, C_{32} = 11, C_{33} = 7$.

$$\text{adj}(A) = \begin{pmatrix} 7 & 1 & -3 \\ -19 & -1 & 11 \\ -11 & -1 & 7 \end{pmatrix} \implies A^{-1} = \frac{1}{4}\begin{pmatrix} 7 & 1 & -3 \\ -19 & -1 & 11 \\ -11 & -1 & 7 \end{pmatrix}$$

$$X = A^{-1}B = \frac{1}{4}\begin{pmatrix} 7(7) + 1(-5) - 3(12) \\ -19(7) - 1(-5) + 11(12) \\ -11(7) - 1(-5) + 7(12) \end{pmatrix} = \frac{1}{4}\begin{pmatrix} 8 \\ 4 \\ 12 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$$

$$\boxed{x = 2,\ y = 1,\ z = 3}$$

*Check:* $2 - 1 + 2(3) = 7\checkmark$, $3(2) + 4(1) - 5(3) = -5\checkmark$, $2(2) - 1 + 3(3) = 12\checkmark$

**Question 3.** If $A$ is a square matrix satisfying $A'A = I$, where $A'$ is the transpose of $A$, then show that $|A| = \pm 1$.

**Solution:** Taking determinants of both sides of $A'A = I$:

$$|A'A| = |I| = 1$$

Using $|A'A| = |A'||A|$ and $|A'| = |A|$:

$$|A||A| = 1 \implies |A|^2 = 1$$

$$\boxed{|A| = \pm 1}$$

**Question 4.** If $A$ is a non-singular square matrix of order 3, prove that $|\text{adj } A| = |A|^2$.

**Solution:** We know $A \cdot (\text{adj } A) = |A|\, I$. Taking the determinant of both sides:

$$|A \cdot \text{adj } A| = \big||A|\, I\big|$$

$$|A| \cdot |\text{adj } A| = |A|^3 \cdot |I| \qquad (\text{since } I \text{ is } 3 \times 3)$$

$$|A| \cdot |\text{adj } A| = |A|^3$$

Since $A$ is non-singular, $|A| \neq 0$, so dividing both sides by $|A|$:

$$\boxed{|\text{adj } A| = |A|^2}$$

**Question 5.** Prove that the determinant $\begin{vmatrix} x & \sin\theta & \cos\theta \\ -\sin\theta & -x & 1 \\ \cos\theta & 1 & x \end{vmatrix}$ is independent of $\theta$.

**Solution:** Expanding along Row 1:

$$\begin{aligned}
\Delta &= x\begin{vmatrix} -x & 1 \\ 1 & x \end{vmatrix} - \sin\theta\begin{vmatrix} -\sin\theta & 1 \\ \cos\theta & x \end{vmatrix} + \cos\theta\begin{vmatrix} -\sin\theta & -x \\ \cos\theta & 1 \end{vmatrix} \\
&= x(-x^2 - 1) - \sin\theta(-x\sin\theta - \cos\theta) + \cos\theta(-\sin\theta + x\cos\theta) \\
&= -x^3 - x + x\sin^2\theta + \sin\theta\cos\theta - \sin\theta\cos\theta + x\cos^2\theta \\
&= -x^3 - x + x(\sin^2\theta + \cos^2\theta) \\
&= -x^3 - x + x \\
&= \boxed{-x^3}
\end{aligned}$$

Since the result does not involve $\theta$, the determinant is independent of $\theta$.

**Question 6.** Using properties of determinants, prove that $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix} = (a-b)(b-c)(c-a)(a+b+c)$.

**Solution:** Apply $R_2 \to R_2 - R_1 \cdot 0$ (not needed here); instead apply column operations $C_2 \to C_2 - C_1$, $C_3 \to C_3 - C_1$ directly does not simplify Row 1 nicely, so we use $R_2 \to R_2$ as is and instead subtract columns:

$$C_1 \to C_1 - C_2, \quad C_2 \to C_2 - C_3$$

$$\Delta = \begin{vmatrix} 0 & 0 & 1 \\ a-b & b-c & c \\ a^3-b^3 & b^3-c^3 & c^3 \end{vmatrix}$$

Take $(a-b)$ common from Column 1 and $(b-c)$ common from Column 2 (using $a^3 - b^3 = (a-b)(a^2+ab+b^2)$ etc.):

$$\Delta = (a-b)(b-c)\begin{vmatrix} 0 & 0 & 1 \\ 1 & 1 & c \\ a^2+ab+b^2 & b^2+bc+c^2 & c^3 \end{vmatrix}$$

Expanding along Row 1 (only the last entry, in position $(1,3)$, is non-zero):

$$\Delta = (a-b)(b-c)\Big[1 \cdot \big\{(b^2+bc+c^2) - (a^2+ab+b^2)\big\}\Big]$$

$$= (a-b)(b-c)\big[bc + c^2 - a^2 - ab\big] = (a-b)(b-c)\big[(c-a)(c+a) + b(c-a)\big]$$

$$= (a-b)(b-c)(c-a)(a+b+c)$$

$$= \boxed{(a-b)(b-c)(c-a)(a+b+c)}$$

*(Verified numerically: $a = 0, b = 1, c = 2$ gives $\Delta = 6$ and $(a-b)(b-c)(c-a)(a+b+c) = (-1)(-1)(2)(3) = 6$.)*

# Quick Revision – Formula Sheet

## Everything from Chapter 4 in one place

1. **Order 2:** $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$
2. **Order 3:** expand along any row/column using cofactors $C_{ij} = (-1)^{i+j} M_{ij}$.
3. **Row/column swap** reverses the sign of the determinant.
4. **Identical rows/columns** $\Rightarrow$ determinant $= 0$.
5. **Common factor** in a row/column can be taken outside the determinant.
6. $R_i \to R_i + kR_j$ leaves the determinant unchanged – the main tool for simplifying before expansion.
7. **Area of a triangle:** $\Delta = \dfrac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$; points are collinear when this determinant is 0.
8. **Adjoint:** transpose of the cofactor matrix; $A(\operatorname{adj} A) = |A|\, I$.
9. **Inverse:** $A^{-1} = \dfrac{1}{|A|}\operatorname{adj}(A)$, exists only if $|A| \neq 0$ (non-singular matrix).
10. **System $AX = B$:** unique solution $X = A^{-1}B$ when $|A| \neq 0$.
11. **Consistency rules:** if $|A| \neq 0$ the system has a unique solution; if $|A| = 0$ and $(\operatorname{adj} A)B \neq 0$ the system is inconsistent (no solution); if $|A| = 0$ and $(\operatorname{adj} A)B = 0$ the system has infinitely many solutions or is inconsistent, requiring further check.

## End of Chapter 4: Determinants

All exercises (4.1 – 4.6) and the Miscellaneous Exercise are covered above with complete, verified, step-by-step solutions.