NCERT Solutions

Class 12 Mathematics

Chapter 5: Continuity and Differentiability

Complete Step-by-Step Solutions – Exercises 5.1 to 5.8

Covers: continuity of functions, algebra of continuous functions, differentiability, chain rule, implicit differentiation, derivatives of inverse trigonometric functions, exponential and logarithmic differentiation, parametric differentiation, second-order derivatives, and Rolle's & Mean Value Theorems. The Miscellaneous Exercise will be provided as a follow-up.

Chapter 5: Continuity and Differentiability

Exercise 5.1 – Continuity (Q1–34)

Q1.

Prove that $f(x) = 5x - 3$ is continuous at $x = 0$, $x = -3$ and $x = 5$.

Solution: f is a polynomial, so it is defined and continuous everywhere. At $x=0$: $f(0)=-3$, and $\lim_{x\to 0} f(x) = -3$ — equal. At $x=-3$: $f(-3)=-18 = \lim_{x\to -3} f(x)$. At $x=5$: $f(5)=22 = \lim_{x\to 5} f(x)$. In every case, $\lim f(x) = f(\text{point})$, so f is continuous at all three points.

Q2.

Examine the continuity of $f(x) = 2x^2 - 1$ at $x = 3$.

Solution: $f(3) = 2(9)-1 = 17$. $\lim_{x\to 3} (2x^2-1) = 2(9)-1 = 17$. Since $f(3) = \lim_{x\to 3}f(x)$, f is continuous at $x=3$.

Q3.

Examine the following for continuity: (a) $f(x)=x-5$ (b) $f(x)=1/(x-5)$, $x\neq 5$ (c) $f(x)=(x^2-25)/(x+5)$, $x\neq -5$ (d) $f(x)=|x-5|$

Solution: (a) A polynomial function, continuous for every real x.
(b) A quotient of two continuous functions (1 and $x-5$) whose denominator is never zero on its domain ($x\neq 5$) $\Rightarrow$ continuous throughout its domain.
(c) Simplifies to $x-5$ for $x\neq -5$ (cancel the common factor $x+5$), which is a polynomial $\Rightarrow$ continuous throughout its domain.
(d) The modulus of the continuous function $x-5$; since the modulus function is continuous everywhere, this composition is continuous everywhere.

Q4.

Prove that $f(x) = x^n$ is continuous at $x = n$, where n is a positive integer.

Solution: f is a polynomial function, hence defined and continuous at every real number, including $x=n$. So $\lim_{x\to n} x^n = n^n = f(n)$, proving continuity at $x=n$.

Q5.

Is f defined by $f(x) = \{x, \text{ if } x\leq 1;\ 5, \text{ if } x>1\}$ continuous at $x=0$? At $x=1$? At $x=2$?

Solution: At x=0: $f(0)=0 = \lim_{x\to 0}x \Rightarrow$ continuous.
At x=1: LHL = $\lim_{x\to 1^-} x = 1$. RHL = $\lim_{x\to 1^+} 5 = 5$. LHL≠RHL $\Rightarrow$ discontinuous.
At x=2: $f(2)=5 = \lim_{x\to 2}5 \Rightarrow$ continuous.

Q6.

Find all points of discontinuity of $f(x) = \{2x+3,\ x\leq 2;\ 2x-3,\ x>2\}$.

Solution: Both pieces are polynomials, so continuity can only fail at $x=2$. LHL = $2(2)+3=7$. RHL = $2(2)-3=1$. LHL≠RHL, so f is discontinuous only at $x=2$.

Q7.

Find all points of discontinuity of $f(x) = \{|x|+3,\ x\leq -3;\ -2x,\ -3<x<3;\ 6x+2,\ x\geq 3\}$.

Solution: At $x=-3$: LHL$=\lim(|x|+3)=6$, RHL$=\lim(-2x)=6$, $f(-3)=6$ — continuous.
At $x=3$: LHL$=\lim(-2x)=-6$, RHL$=\lim(6x+2)=20$ — unequal, so discontinuous at $x=3$. Elsewhere the pieces are polynomials, hence continuous.

Only point of discontinuity: x = 3.

Q8.

Find all points of discontinuity of $f(x) = |x|/x$, $x \neq 0$; $f(0)=0$.

Solution: For $x>0$, $f(x)=1$; for $x<0$, $f(x)=-1$. At $x=0$: LHL$=-1$, RHL$=1$, $f(0)=0$ — all different, so discontinuous at $x=0$. Continuous everywhere else.

Q9.

Find all points of discontinuity of $f(x) = x/|x|$, $x<0$; $f(x)=-1$, $x \geq 0$.

Solution: For $x<0$, $|x|=-x$, so $x/|x| = -1$. Thus $f(x) = -1$ for every real x (both branches agree) — f is simply the constant function -1, which is continuous everywhere.

Q10.

Find all points of discontinuity of $f(x) = \{x+1,\ x \geq 1;\ x^2+1,\ x<1\}$.

Solution: At $x=1$: LHL$=\lim_{x\to 1^-}(x^2+1)=2$, RHL$=\lim_{x\to 1^+}(x+1)=2$, $f(1)=2$ — equal. Both pieces are polynomials, so f is continuous everywhere.

Q11.

Find all points of discontinuity of $f(x) = \{x^3-3,\ x \leq 2;\ x^2+1,\ x>2\}$.

Solution: At $x=2$: LHL$=2^3-3=5$, RHL$=2^2+1=5$, $f(2)=5$ — equal, so continuous at $x=2$ and everywhere (polynomial pieces).

Q12.

Find all points of discontinuity of $f(x) = \{x^{10}-1,\ x \leq 1;\ x^2,\ x>1\}$.

Solution: At $x=1$: LHL$=1-1=0$. RHL$=1^2=1$. LHL$\neq$RHL, so f is discontinuous at $x=1$ (the only point of discontinuity).

Q13.

Is $f(x) = \{x+5,\ x \leq 1;\ x-5,\ x>1\}$ a continuous function?

Solution: At $x=1$: LHL$=1+5=6$, RHL$=1-5=-4$. LHL$\neq$RHL, so f is discontinuous at $x=1$; hence f is not a continuous function.

Q14.

Discuss the continuity of $f(x) = \{3,\ 0 \leq x \leq 1;\ 4,\ 1<x<3;\ 5,\ 3 \leq x \leq 10\}$.

Solution: At $x=1$: LHL$=3$, RHL$=4$ — unequal, discontinuous.
At $x=3$: LHL$=4$, RHL$=5$ — unequal, discontinuous.
f is continuous on the open intervals $(0,1)$, $(1,3)$, $(3,10)$, but discontinuous at $x=1$ and $x=3$.

Q15.

Discuss the continuity of $f(x) = \{2x,\ x<0;\ 0,\ 0 \leq x \leq 1;\ 4x,\ x>1\}$.

Solution: At $x=0$: LHL$=\lim_{x\to 0^-}2x=0$, RHL$=0$ (from middle branch), $f(0)=0$ — continuous.
At $x=1$: LHL$=0$ (middle branch), RHL$=\lim_{x\to 1^+}4x=4$, $f(1)=0$ — LHL$\neq$RHL, discontinuous.

Only point of discontinuity: $x = 1$.

Q16.

Discuss the continuity of $f(x) = \{-2,\ x \leq -1;\ 2x,\ -1<x \leq 1;\ 2,\ x>1\}$.

Solution: At $x=-1$: LHL$=-2$, RHL$=\lim_{x\to -1^+}2x=-2$, $f(-1)=-2$ — continuous.
At $x=1$: LHL$=\lim_{x\to 1^-}2x=2$, RHL$=2$, $f(1)=2$ — continuous. So f is continuous everywhere.

Q17.

Find the relation between a and b so that $f(x)=\{ax+1, x\leq 3; bx+3, x>3\}$ is continuous at $x=3$.

Solution: $f(3)=3a+1$. LHL=$3a+1$. RHL=$3b+3$. For continuity, LHL=RHL=$f(3)$: $3a+1=3b+3$, i.e. $3a - 3b = 2$.

Relation: $3a - 3b = 2$ (equivalently $a - b = 2/3$).

Q18.

For what value of λ is $f(x)=\{\lambda(x^2-2x), x\leq 0; 4x+1, x>0\}$ continuous at $x=0$? What about continuity at $x=1$?

Solution: $f(0)=\lambda(0)=0$. LHL=$\lim_{x\to 0^-}\lambda(x^2-2x)=0$. RHL=$\lim_{x\to 0^+}(4x+1)=1$. Since LHL=0 but RHL=1 regardless of λ, they can never be equal.

No value of λ makes f continuous at $x=0$. At $x=1$, $f(x)=4x+1$ (a polynomial) throughout a neighbourhood of 1, so f is continuous at $x=1$ for every value of λ.

Q19.

Show that the function $g(x) = x - [x]$ (fractional part function, $[x]$ = greatest integer $\leq x$) is discontinuous at every integer.

Solution: Let n be an integer. LHL = $\lim_{x\to n^-}(x-[x]) = n-(n-1) = 1$ (just left of n, $[x]=n-1$). RHL = $\lim_{x\to n^+}(x-[x]) = n-n = 0$. $g(n)=n-n=0$. Since LHL$\neq$RHL, g is discontinuous at every integer n.

Q20.

Is the function $f(x) = x^2 - \sin x + 5$ continuous at $x = \pi$?

Solution: x^2+5 (polynomial) and $\sin x$ are both continuous everywhere; their combination (sum/difference of continuous functions) is continuous everywhere, so f is continuous at $x=\pi$, with $f(\pi)=\pi^2+5$.

Q21.

Discuss the continuity of: (a) $\sin x + \cos x$ (b) $\sin x - \cos x$ (c) $\sin x \cdot \cos x$.

Solution: $\sin x$ and $\cos x$ are each continuous on all of R (standard result). By the algebra of continuous functions, their sum, difference and product are also continuous everywhere. Hence all three functions (a), (b), (c) are continuous on R.

Q22.

Discuss the continuity of the cosine, cosecant, secant and cotangent functions.

Solution: $\cos x$ is continuous everywhere (like $\sin x$). $\sec x = 1/\cos x$ is continuous wherever $\cos x\neq 0$, i.e. everywhere except $x=(2n+1)\pi/2$. $\text{cosec}\, x = 1/\sin x$ is continuous wherever $\sin x\neq 0$, i.e. except $x=n\pi$. $\cot x = \cos x/\sin x$ is continuous except at $x=n\pi$. (n any integer)

Q23.

Find all points of discontinuity of $f(x) = \{\sin x / x, x<0; x+1, x\geq 0\}$.

Solution: At $x=0$: LHL = $\lim_{x\to 0^-} \sin x/x = 1$ (standard limit). RHL = $\lim_{x\to 0^+}(x+1)=1$. $f(0)=1$. All equal $\Rightarrow$ continuous at 0. Both pieces are continuous on their domains, so f has no point of discontinuity.

Q24.

Determine if $f(x) = \{x^2\sin(1/x), x\neq 0; 0, x=0\}$ is continuous at $x=0$.

Solution: Since $|\sin(1/x)|\leq 1$, we have $|x^2\sin(1/x)| \leq x^2 \to 0$ as $x\to 0$. By the Squeeze theorem, $\lim_{x\to 0} f(x)=0=f(0)$. So f is continuous at $x=0$.

Q25.

Discuss the continuity of $f(x) = \{\sin x - \cos x, x\neq 0; -1, x=0\}$ at $x = 0$.

Solution: $\lim_{x\to 0}(\sin x-\cos x) = 0-1 = -1 = f(0)$. So f is continuous at $x=0$.

Q26.

Find the value of k so that $f(x) = \{k \cos x/(\pi-2x),\ x\neq\pi/2;\ 3,\ x=\pi/2\}$ is continuous at $x=\pi/2$.

Solution: Put $x=\pi/2+h$, $h\to0$: $\pi-2x=-2h$, $\cos x=\cos(\pi/2+h)=-\sin h$. So $k \cos x/(\pi-2x)=k(-\sin h)/(-2h)=k(\sin h/h)/2 \to k/2$ as $h\to0$. Setting this equal to $f(\pi/2)=3$: $k/2=3$.

k = 6.

Q27.

Find k so that $f(x)=\{kx^2,\ x\leq2;\ 3,\ x>2\}$ is continuous at $x=2$.

Solution: LHL = $f(2) = k(4)=4k$. RHL=3. Setting $4k=3$.

k = 3/4.

Q28.

Find k so that $f(x)=\{kx+1,\ x\leq\pi;\ \cos x,\ x>\pi\}$ is continuous at $x=\pi$.

Solution: LHL = $f(\pi) = k\pi+1$. RHL = $\lim_{x\to\pi^+}\cos x = \cos\pi=-1$. Setting $k\pi+1=-1 \Rightarrow k\pi=-2$.

k = $-2/\pi$.

Q29.

Find k so that $f(x)=\{kx+1,\ x\leq5;\ 3x-5,\ x>5\}$ is continuous at $x=5$.

Solution: LHL=$f(5)=5k+1$. RHL=$3(5)-5=10$. Setting $5k+1=10 \Rightarrow 5k=9$.

k = 9/5.

Q30.

Find a and b so that $f(x)=\{5,\ x\leq2;\ ax+b,\ 2<x<10;\ 21,\ x\geq10\}$ is continuous.

Solution: At $x=2$: LHL=5, RHL=$2a+b \Rightarrow 2a+b=5$ …(i)
At $x=10$: LHL=$10a+b$, RHL=$21 \Rightarrow 10a+b=21$ …(ii)
Subtracting (i) from (ii): $8a=16 \Rightarrow a=2$. Substituting into (i): $4+b=5 \Rightarrow b=1$.

a = 2, b = 1.

Q31.

Show that $f(x) = \cos(x^2)$ is a continuous function.

Solution: x^2 is continuous everywhere (polynomial), and cos t is continuous everywhere. The composition of two continuous functions is continuous, so $f(x)=\cos(x^2)$ is continuous on all of R.

Q32.

Show that $f(x) = |\cos x|$ is a continuous function.

Solution: cos x is continuous on R, and the modulus function |t| is continuous everywhere. Their composition |cos x| is therefore continuous on R.

Q33.

Examine the continuity of $f(x) = \sin|x|$.

Solution: |x| is continuous everywhere, and sin t is continuous everywhere, so the composition sin|x| is continuous on all of R.

Q34.

Discuss the continuity of $f(x) = |x| - |x+1|$.

Solution: $|x|$ and $|x+1|$ are each continuous everywhere (modulus of a continuous linear function). The difference of two continuous functions is continuous, so f is continuous on all of R.

Exercise 5.2 – Differentiability & Chain Rule (Q1–10)

Q1.

Differentiate $\sin(x^2+5)$ with respect to x.

Solution: By the chain rule: $d/dx[\sin(x^2+5)] = \cos(x^2+5)\cdot 2x$.

Answer: $2x\cos(x^2+5)$

Q2.

Differentiate $\cos(\sin x)$ with respect to x.

Solution: $d/dx[\cos(\sin x)] = -\sin(\sin x)\cdot\cos x$.

Answer: $-\cos x \cdot \sin(\sin x)$

Q3.

Differentiate $\sin(ax+b)$ with respect to x.

Solution: $d/dx[\sin(ax+b)] = a\cos(ax+b)$, since $d/dx(ax+b)=a$.

Answer: $a\cos(ax+b)$

Q4.

Differentiate $\sec(\tan(\sqrt{x}))$ with respect to x.

Solution: Let $u=\sqrt{x}$. $d/dx[\sec(\tan u)] = \sec(\tan u)\tan(\tan u)\cdot\sec^2 u\cdot du/dx$, and $du/dx=1/(2\sqrt{x})$.

Answer: $[\sec(\tan\sqrt{x}) \cdot \tan(\tan\sqrt{x}) \cdot \sec^2\sqrt{x}] / (2\sqrt{x})$

Q5.

Differentiate $\sin(ax+b)/\cos(cx+d)$ with respect to x.

Solution: Quotient rule with $u=\sin(ax+b)$, $v=\cos(cx+d)$: $u'=a\cos(ax+b)$, $v'=-c\sin(cx+d)$.
$dy/dx = (u'v-uv')/v^2 = [a\cos(ax+b)\cos(cx+d) + c\sin(ax+b)\sin(cx+d)] / \cos^2(cx+d)$.

Q6.

Differentiate $\cos(x^3)\cdot\sin^2(x\&sup5;)$ with respect to x.

Solution: Product rule. $d/dx[\cos(x^3)] = -3x^2\sin(x^3)$. $d/dx[\sin^2(x\&sup5;)] = 2\sin(x\&sup5;)\cos(x\&sup5;)\cdot 5x\blacksquare = 5x\blacksquare\sin(2x\blacksquare)$.
$dy/dx = -3x^2\sin(x^3)\sin^2(x\blacksquare) + 5x\blacksquare\cos(x^3)\sin(2x\blacksquare)$.

Q7.

Differentiate $2\sqrt{(\cot(x^2))}$ with respect to x.

Solution: $dy/dx = 2\cdot[1/(2\sqrt{(\cot(x^2))})]\cdot(-\mathrm{cosec}^2(x^2))\cdot 2x = -2x\,\mathrm{cosec}^2(x^2)/\sqrt{(\cot(x^2))}$.

Q8.

Differentiate $\cos(\sqrt{x})$ with respect to x.

Solution: $dy/dx = -\sin(\sqrt{x})\cdot 1/(2\sqrt{x})$.

Answer: $-\sin(\sqrt{x}) / (2\sqrt{x})$

Q9.

Prove that $f(x)=|x-1|$, $x\in R$, is not differentiable at $x=1$.

Solution: Left-hand derivative: $\lim_{h\to 0^-} [f(1+h)-f(1)]/h = \lim_{h\to 0^-} |h|/h = \lim (-h)/h = -1$ (since $h<0$, $|h|=-h$).
Right-hand derivative: $\lim_{h\to 0^+} |h|/h = \lim h/h = 1$ (since $h>0$, $|h|=h$).

Since LHD = $-1 \neq 1$ = RHD, f is not differentiable at $x=1$.

Q10.

Prove that the greatest integer function $f(x)=[x]$, $0<x<3$, is not differentiable at $x=1$ and $x=2$. (Standard companion result also proved: $f(x)=x^n\sin(1/x)$, $x\neq0$, $f(0)=0$, is continuous but not differentiable at 0 for $0<n\leq1$, and is differentiable at 0 for $n>1$.)

Solution: [x] at x=1,2: at $x=1$, LHL of $[x]=0$, RHL$=1$, so $[x]$ itself is discontinuous at $x=1$ (similarly at $x=2$), and a function that is not even continuous at a point cannot be differentiable there.

General result: For $f(x)=x^n\sin(1/x)$ $(x\neq0)$, $f(0)=0$: since $|x^n\sin(1/x)| \leq |x|^n \to 0$ $(n>0)$, f is continuous at 0. The derivative at 0 is $\lim_{h\to0} h^n\sin(1/h)/h = \lim_{h\to0} h^{n-1}\sin(1/h)$. This limit exists (and equals 0) only when $n-1>0$, i.e. $n>1$, because $\sin(1/h)$ oscillates boundedly while $h^{n-1}\to0$. For $0<n\leq1$, h^{n-1} does not tend to 0, so the limit fails to exist.

Hence f is differentiable at 0 only for $n > 1$; for $0 < n \leq 1$ it is continuous but not differentiable at 0.

Exercise 5.3 – Implicit Differentiation (Q1–15)

Q1.

Find dy/dx: $2x + 3y = \sin y$

Solution: Differentiating both sides w.r.t. x: $2 + 3(dy/dx) = \cos y\,(dy/dx)$
$2 = (dy/dx)(\cos y - 3)$

$dy/dx = 2/(\cos y - 3) = -2/(3 - \cos y)$

Q2.

Find dy/dx: $x^2 + xy + y^2 = 100$

Solution: $2x + (y + x\cdot dy/dx) + 2y\cdot dy/dx = 0$
$(2x+y) + dy/dx(x+2y) = 0$

$dy/dx = -(2x+y)/(x+2y)$

Q3.

Find dy/dx: $x^3 + x^2y + xy^2 + y^3 = 81$

Solution: $3x^2 + (2xy+x^2y') + (y^2+2xy\cdot y') + 3y^2y' = 0$
$(3x^2+2xy+y^2) + y'(x^2+2xy+3y^2) = 0$

$dy/dx = -(3x^2+2xy+y^2) / (x^2+2xy+3y^2)$

Q4.

Find dy/dx: $\sin^2 y + \cos(xy) = k$ (k constant)

Solution: $2 \sin y \cos y \cdot y' - \sin(xy)\cdot(y + xy') = 0$
$y'[2 \sin y\cos y - x \sin(xy)] = y \sin(xy)$

$dy/dx = y \sin(xy) / [\sin 2y - x \sin(xy)]$

Q5.

Find dy/dx: $\sin^2 x + \cos^2 y = 1$

Solution: $2 \sin x \cos x + 2\cos y(-\sin y)y' = 0$
$\sin 2x = \sin 2y \cdot y'$

$dy/dx = \sin 2x / \sin 2y$

Q6.

Find dy/dx: $y = \tan^{-1}[(3x-x^3)/(1-3x^2)]$, $-1/\sqrt{3} < x < 1/\sqrt{3}$

Solution: Put $x = \tan\theta$. Then $(3x-x^3)/(1-3x^2) = \tan 3\theta$ (standard triple-angle identity), so $y = \tan^{-1}(\tan 3\theta) = 3\theta = 3\tan^{-1}x$.

$dy/dx = 3/(1+x^2)$

Q7.

Find dy/dx: $y = \tan^{-1}[(\sqrt{(1+x^2)}-1)/x]$, $x\neq 0$

Solution: Put $x=\tan\theta$. $\sqrt{(1+x^2)}=\sec\theta$. $(\sec\theta-1)/\tan\theta = (1-\cos\theta)/\sin\theta = \tan(\theta/2)$. So $y=\theta/2 = (1/2)\tan^{-1}x$.

$dy/dx = 1/[2(1+x^2)]$

Q8.

Find dy/dx: $y = \tan^{-1}[x/(\sqrt{(1-x^2)})]$, $|x|<1$

Solution: Put $x=\sin\theta$. $\sqrt{1-x^2}=\cos\theta$. $x/\sqrt{1-x^2}=\tan\theta$. So $y=\theta=\sin^{-1}x$.

$dy/dx = 1/\sqrt{1-x^2}$

Q9.

Find dy/dx: $y = \cos^{-1}[(1-x^2)/(1+x^2)]$, $0<x<1$

Solution: Put $x=\tan\theta$. $(1-x^2)/(1+x^2)=\cos 2\theta$. So $y=\cos^{-1}(\cos 2\theta)=2\theta=2\tan^{-1}x$.

$dy/dx = 2/(1+x^2)$

Q10.

Find dy/dx: $y = \sin^{-1}[(1-x^2)/(1+x^2)]$, $0<x<1$

Solution: Put $x=\tan\theta$. $(1-x^2)/(1+x^2)=\cos 2\theta=\sin(\pi/2-2\theta)$. So $y=\pi/2-2\theta=\pi/2-2\tan^{-1}x$.

$dy/dx = -2/(1+x^2)$

Q11.

Find dy/dx: $y = \cos^{-1}[2x/(1+x^2)]$, $-1<x<1$

Solution: Direct differentiation: $1 - [2x/(1+x^2)]^2 = (1-x^2)^2/(1+x^2)^2$, so $\sqrt{(\dots)} = (1-x^2)/(1+x^2)$ for $|x|<1$. Also $d/dx[2x/(1+x^2)] = 2(1-x^2)/(1+x^2)^2$.
$dy/dx = -1/[(1-x^2)/(1+x^2)] \cdot 2(1-x^2)/(1+x^2)^2$

$dy/dx = -2/(1+x^2)$

Q12.

Find dy/dx: $y = \sin^{-1}[2x\sqrt{1-x^2}]$, $-1/\sqrt{2} < x < 1/\sqrt{2}$

Solution: Put $x=\sin\theta$. $2x\sqrt{1-x^2} = 2\sin\theta\cos\theta = \sin 2\theta$. So $y=2\theta=2\sin^{-1}x$.

$dy/dx = 2/\sqrt{1-x^2}$

Q13.

Find dy/dx: $y = \sec^{-1}[1/(2x^2-1)]$, $0<x<1/\sqrt{2}$

Solution: Put $x=\cos\theta$. $2x^2-1=\cos 2\theta$, so $1/(2x^2-1)=\sec 2\theta$. $y = \sec^{-1}(\sec 2\theta)=2\theta=2\cos^{-1}x$.

$dy/dx = -2/\sqrt{1-x^2}$

Q14.

If $x\sqrt{1+y} + y\sqrt{1+x} = 0$, for $x\neq y$, prove that $dy/dx = -1/(1+x)^2$.

Solution: Rearranging: $x\sqrt{1+y} = -y\sqrt{1+x}$. Squaring: $x^2(1+y) = y^2(1+x)$
$x^2+x^2y - y^2 - xy^2 = 0 \Rightarrow (x^2-y^2) + xy(x-y) = 0 \Rightarrow (x-y)(x+y) + xy(x-y) = 0$
Since $x\neq y$, divide by $(x-y)$: $x+y+xy = 0 \Rightarrow y(1+x) = -x \Rightarrow y = -x/(1+x)$
Differentiating: $dy/dx = -[(1+x)\cdot 1 - x\cdot 1]/(1+x)^2 = -1/(1+x)^2$. Proved.

Q15.

If $(x-a)^2 + (y-b)^2 = c^2$, for some $c>0$, prove that $[1+(dy/dx)^2]^{3/2} \div (d^2y/dx^2)$ is a constant independent of a and b.

Solution: Differentiating: $2(x-a) + 2(y-b)y' = 0 \Rightarrow y' = -(x-a)/(y-b)$.
Differentiating again: $y'' = -[(y-b) - (x-a)y']/(y-b)^2 = -[(y-b)^2+(x-a)^2]/(y-b)^3 = -c^2/(y-b)^3$ (using the original equation).
$1+(y')^2 = [(y-b)^2+(x-a)^2]/(y-b)^2 = c^2/(y-b)^2$, so $[1+(y')^2]^{3/2} = c^3/|y-b|^3$.
Dividing by y'': $[c^3/(y-b)^3] \div [-c^2/(y-b)^3] = -c$, which depends only on c, not on a or b. Proved.

Exercise 5.4 – Exponential & Logarithmic Derivatives (Q1–10)

Q1.

Differentiate $e^x/\sin x$ with respect to x.

Solution: Quotient rule: $dy/dx = [e^x \sin x - e^x \cos x]/\sin^2 x$.

Answer: $e^x(\sin x - \cos x)/\sin^2 x$

Q2.

Differentiate $e^{\sin^{-1} x}$ with respect to x.

Answer: $e^{\sin^{-1} x} / \sqrt{(1-x^2)}$ (by the chain rule)

Q3.

Differentiate e^{x^3} with respect to x.

Answer: $3x^2 e^{x^3}$

Q4.

Differentiate $\sin(\tan^{-1} e^{-x})$ with respect to x.

Solution: Chain rule: $dy/dx = \cos(\tan^{-1} e^{-x}) \cdot [1/(1+e^{-2x})] \cdot (-e^{-x})$.

Answer: $-e^{-x} \cos(\tan^{-1} e^{-x}) / (1+e^{-2x})$

Q5.

Differentiate $\log(\cos(e^x))$ with respect to x.

Solution: $dy/dx = [1/\cos(e^x)] \cdot (-\sin(e^x)) \cdot e^x = -e^x \tan(e^x)$.

Q6.

Differentiate $e^x + e^{x^2} + e^{x^3} + \ldots + e^{x^{\blacksquare}}$ with respect to x.

Answer: $e^x + 2x\, e^{x^2} + 3x^2 e^{x^3} + 4x^3 e^{x^{\blacksquare}} + 5x^{\blacksquare} e^{x^{\blacksquare}}$

Q7.

Differentiate $\sqrt{(e^{\sqrt{x}})}$, $x>0$, with respect to x.

Solution: $\sqrt{(e^{\sqrt{x}})} = e^{\sqrt{x}/2}$. $dy/dx = e^{\sqrt{x}/2} \cdot (1/2) \cdot 1/(2\sqrt{x})$.

Answer: $\sqrt{(e^{\sqrt{x}})} / (4\sqrt{x})$

Q8.

Differentiate $\log(\log x)$, $x>1$, with respect to x.

Solution: $dy/dx = [1/\log x] \cdot (1/x)$.

Answer: $1/(x \log x)$

Q9.

Differentiate $\cos x / \log x$, $x>0$, $x \neq 1$, with respect to x.

Solution: Quotient rule: $dy/dx = [-\sin x \cdot \log x - \cos x \cdot (1/x)] / (\log x)^2$.

Answer: $-[x \sin x \log x + \cos x] / [x(\log x)^2]$

Q10.

Differentiate $\cos(\log x + e^x)$, $x>0$, with respect to x.

Answer: $-\sin(\log x + e^x) \cdot (1/x + e^x)$

Exercise 5.5 – Logarithmic Differentiation (Q1–18)

Q1.

Differentiate cos x · cos 2x · cos 3x with respect to x.

Solution: Take log: log y = log cosx + log cos2x + log cos3x.
y'/y = –tanx – 2tan2x – 3tan3x.

dy/dx = –cosx·cos2x·cos3x · (tanx + 2tan2x + 3tan3x)

Q2.

Differentiate √[(x–1)(x–2)] / [(x–3)(x–4)(x–5)] with respect to x.

Solution: log y = ½[log(x–1)+log(x–2)] – [log(x–3)+log(x–4)+log(x–5)]
y'/y = ½[1/(x–1)+1/(x–2)] – [1/(x–3)+1/(x–4)+1/(x–5)]

dy/dx = y · {½[1/(x–1)+1/(x–2)] – [1/(x–3)+1/(x–4)+1/(x–5)]}, with y as given above

Q3.

Differentiate $(\log x)^{\cos x}$ with respect to x.

Solution: log y = cosx · log(logx). y'/y = –sinx·log(logx) + cosx/(x logx).

dy/dx = $(\log x)^{\cos x}$ · [cosx/(x logx) – sinx·log(logx)]

Q4.

Differentiate $x^x - 2^{\sin x}$ with respect to x.

Solution: For $u=x^x$: logu=xlogx, u'/u=logx+1, so $u'=x^x(1+\log x)$.
For $v=2^{\sin x}$: logv=sinx·log2, v'/v=cosx log2, so $v'=2^{\sin x}\cos x \log 2$.

dy/dx = $x^x(1+\log x) - 2^{\sin x}\cos x\cdot\log 2$

Q5.

Differentiate (x+3)²(x+4)³(x+5)■ with respect to x.

Solution: log y = 2log(x+3)+3log(x+4)+4log(x+5). y'/y = 2/(x+3)+3/(x+4)+4/(x+5).

dy/dx = (x+3)²(x+4)³(x+5)■ · [2/(x+3)+3/(x+4)+4/(x+5)]

Q6.

Differentiate $(x+1/x)^x + x^{1+1/x}$ with respect to x.

Solution: For $u=(x+1/x)^x$: logu = x log(x+1/x). u'/u = log(x+1/x) + (x²–1)/(x²+1) (after simplification), so $u'=(x+1/x)^x[\log(x+1/x)+(x^2-1)/(x^2+1)]$.
For $v=x^{1+1/x}$: logv=(1+1/x)logx. v'/v = –logx/x² + 1/x + 1/x², so $v' = x^{1+1/x}[1/x + (1-\log x)/x^2]$.

dy/dx = u' + v' as derived above

Q7.

Differentiate $(\log x)^x + x^{\log x}$ with respect to x.

Solution: For $u=(\log x)^x$: logu=x log(logx). u'/u=log(logx)+1/logx, so $u'=(\log x)^x[\log(\log x)+1/\log x]$.
For $v=x^{\log x}$: logv=(logx)². v'/v=2logx/x, so $v'=2x^{\log x-1}\log x$.

dy/dx = $(\log x)^x[\log(\log x)+1/\log x] + 2x^{\log x-1}\log x$

Q8.

Differentiate $(\sin x)^x + \sin^{-1}\sqrt{x}$ with respect to x.

Solution: For $u=(\sin x)^x$: $\log u = x\log(\sin x)$. $u'/u=\log(\sin x)+x\cot x$, so $u'=(\sin x)^x[\log(\sin x)+x\cot x]$.
For $v=\sin^{-1}\sqrt{x}$: $v' = 1/(2\sqrt{(x(1-x))})$.

$dy/dx = (\sin x)^x[\log(\sin x)+x\cot x] + 1/(2\sqrt{(x(1-x))})$

Q9.

Differentiate $x^{\sin x} + (\sin x)^{\cos x}$ with respect to x.

Solution: For $u=x^{\sin x}$: $u'/u=\cos x\cdot\log x + \sin x/x$, so $u'=x^{\sin x}[\cos x\log x+\sin x/x]$.
For $v=(\sin x)^{\cos x}$: $v'/v=-\sin x\log(\sin x)+\cos x\cot x$, so $v'=(\sin x)^{\cos x}[\cos x\cot x-\sin x\log(\sin x)]$.

$dy/dx = x^{\sin x}[\cos x\log x + \sin x/x] + (\sin x)^{\cos x}[\cos x\cot x - \sin x\log(\sin x)]$

Q10.

Differentiate $x^{x\cos x} + (x^2+1)/(x^2-1)$ with respect to x.

Solution: For $u=x^{x\cos x}$: $\log u=x\cos x\cdot\log x$. $u'/u = \cos x\log x - x\sin x\log x + \cos x$, so
$u'=x^{x\cos x}[\cos x(1+\log x)-x\sin x\log x]$.
For $v=(x^2+1)/(x^2-1)$: $v' = [2x(x^2-1)-(x^2+1)(2x)]/(x^2-1)^2 = -4x/(x^2-1)^2$.

$dy/dx = x^{x\cos x}[\cos x(1+\log x)-x\sin x\log x] - 4x/(x^2-1)^2$

Q11.

Differentiate $(x\cos x)^x + (x\sin x)^{1/x}$ with respect to x.

Solution: For $u=(x\cos x)^x$: $\log u=x[\log x+\log\cos x]$. $u'/u=\log x+\log\cos x+1-x\tan x$.
For $v=(x\sin x)^{1/x}$: $\log v=(1/x)[\log x+\log\sin x]$. $v'/v=(1-\log x-\log\sin x)/x^2+\cot x/x$.

$dy/dx = (x\cos x)^x[\log x+\log\cos x+1-x\tan x] + (x\sin x)^{1/x}[(1-\log x-\log\sin x)/x^2+\cot x/x]$

Q12.

Find dy/dx of $x^y + y^x = 1$.

Solution: Differentiating x^y ($\log u=y\log x$): $u'=x^y(y'\log x+y/x)$. Differentiating y^x ($\log v=x\log y$): $v'=y^x(\log y+xy'/y)$.
Sum=0.

$dy/dx = -[(y\cdot x^y)/x + y^x\log y] / [x^y\log x + (x\cdot y^x)/y]$

Q13.

Find dy/dx of $y^x = x^y$.

Solution: Take log: $x\log y = y\log x$. Differentiate: $\log y + xy'/y = y'\log x + y/x$. Rearranging and multiplying by xy:

$dy/dx = y(y - x\log y) / [x(x - y\log x)]$

Q14.

Find dy/dx of $(\cos x)^y = (\cos y)^x$.

Solution: Take log: $y\log(\cos x) = x\log(\cos y)$. Differentiate: $y'\log\cos x - y\tan x = \log\cos y - x\tan y\cdot y'$.

$dy/dx = [\log(\cos y) + y\tan x] / [\log(\cos x) + x\tan y]$

Q15.

Find dy/dx of $xy = e^{(x-y)}$.

Solution: Take log: $\log x+\log y = x-y$. Differentiate: $1/x + y'/y = 1-y'$. Rearranging: $y'[(1+y)/y] = (x-1)/x$.

$dy/dx = y(x-1) / [x(1+y)]$

Q16.

Find dy/dx of $y=(x^2–5x+8)(x^3+7x+9)$, using the product rule; verify by expanding the product first.

Solution: Product rule: $y' = (2x–5)(x^3+7x+9) + (x^2–5x+8)(3x^2+7)$. Expanding both this and the fully-multiplied polynomial before differentiating gives the same final result:

dy/dx = 5x■ – $20x^3 + 45x^2 – 52x + 11$

Q17.

Differentiate $(x^2–5x+8)(x^3+7x+9)$ in three ways: product rule, expansion, and logarithmic differentiation; verify all three give the same answer.

Solution: Logarithmic method: $\log y=\log(x^2–5x+8)+\log(x^3+7x+9)$. $y'=y[(2x–5)/(x^2–5x+8) + (3x^2+7)/(x^3+7x+9)] = (2x–5)(x^3+7x+9)+(x^2–5x+8)(3x^2+7)$, which is exactly the product-rule expression from Q16, and both expand to the same polynomial.

dy/dx = 5x■ – $20x^3 + 45x^2 – 52x + 11$ (all three methods agree)

Q18.

If u, v, w are functions of x, show that d/dx(uvw) = (du/dx)vw + u(dv/dx)w + uv(dw/dx), using logarithmic differentiation.

Solution: Let y=uvw. Taking log: logy = logu+logv+logw. Differentiating: y'/y = u'/u + v'/v + w'/w.
So y' = y(u'/u+v'/v+w'/w) = uvw·u'/u + uvw·v'/v + uvw·w'/w = u'vw + uv'w + uvw'. Proved.

Exercise 5.6 – Parametric Differentiation (Q1–11)

Q1.

Find dy/dx: $x=2at^2$, $y=at$■.

Solution: $dx/dt=4at$, $dy/dt=4at^3$.

$dy/dx = t^2$

Q2.

Find dy/dx: $x=a\cos\theta$, $y=b\cos\theta$.

Solution: $dx/d\theta=-a\sin\theta$, $dy/d\theta=-b\sin\theta$.

$dy/dx = b/a$

Q3.

Find dy/dx: $x=\sin t$, $y=\cos 2t$.

Solution: $dx/dt=\cos t$, $dy/dt=-2\sin 2t=-4\sin t\cos t$.

$dy/dx = -4\sin t$ (i.e. $-4x$)

Q4.

Find dy/dx: $x=4t$, $y=4/t$.

Solution: $dx/dt=4$, $dy/dt=-4/t^2$.

$dy/dx = -1/t^2$

Q5.

Find dy/dx: $x=\cos\theta-\cos 2\theta$, $y=\sin\theta-\sin 2\theta$.

Solution: $dx/d\theta=-\sin\theta+2\sin 2\theta$, $dy/d\theta=\cos\theta-2\cos 2\theta$.

$dy/dx = (\cos\theta-2\cos 2\theta) / (2\sin 2\theta-\sin\theta)$

Q6.

Find dy/dx: $x=a(\theta-\sin\theta)$, $y=a(1+\cos\theta)$.

Solution: $dx/d\theta=a(1-\cos\theta)$, $dy/d\theta=-a\sin\theta$. Using $\sin\theta=2\sin(\theta/2)\cos(\theta/2)$, $1-\cos\theta=2\sin^2(\theta/2)$:

$dy/dx = -\cot(\theta/2)$

Q7.

Find dy/dx: $x=\sin^3 t/\sqrt{\cos 2t}$, $y=\cos^3 t/\sqrt{\cos 2t}$.

Solution: Differentiating carefully (product/chain rule on each, then simplifying using cos2t identities), the expression reduces neatly:

$dy/dx = -\cot 3t$

Q8.

Find dy/dx: $x=a(\cos\theta+\theta\sin\theta)$, $y=a(\sin\theta-\theta\cos\theta)$.

Solution: $dx/d\theta=a[-\sin\theta+\sin\theta+\theta\cos\theta]=a\theta\cos\theta$. $dy/d\theta=a[\cos\theta-\cos\theta+\theta\sin\theta]=a\theta\sin\theta$.

$dy/dx = \tan\theta$

Q9.

Find dy/dx: $x=a\cos t + \log\tan(t/2)$, $y=a\sin t$.

Solution: $dx/dt = -a\sin t + [\sec^2(t/2)/\tan(t/2)]\cdot\frac{1}{2} = -a\sin t + 1/\sin t$ (using $\sin t = 2\sin(t/2)\cos(t/2)$). $dy/dt = a\cos t$.

$dy/dx = a \sin t \cos t / (1 - a \sin^2 t)$

Q10.

Find dy/dx: $x=a(\sin\theta-\theta\cos\theta)$, $y=a(\cos\theta+\theta\sin\theta)$.

Solution: $dx/d\theta=a[\cos\theta-\cos\theta+\theta\sin\theta]=a\theta\sin\theta$. $dy/d\theta=a[-\sin\theta+\sin\theta+\theta\cos\theta]=a\theta\cos\theta$.

$dy/dx = \cot\theta$

Q11.

Find dy/dx: $x=3\cos t - \cos 3t$, $y=3\sin t - \sin 3t$.

Solution: $dx/dt=3(\sin 3t-\sin t)$, $dy/dt=3(\cos t-\cos 3t)$. Using sum-to-product identities: $\cos t-\cos 3t=2\sin 2t \sin t$, $\sin 3t-\sin t=2\cos 2t \sin t$.

$dy/dx = \tan 2t$

Exercise 5.7 – Second Order Derivatives (Q1–17)

Q1.

Find d^2y/dx^2: $y=x^2+3x+2$

$y' = 2x+3$, $y'' = 2$

Q2.

Find d^2y/dx^2: $y=x^{2}$■

$y' = 20x^{1}$■, $y'' = 380x^{1}$■

Q3.

Find d^2y/dx^2: $y=x \cos x$

Solution: $y' = \cos x - x\sin x$. $y'' = -\sin x - \sin x - x\cos x$.

$y'' = -2\sin x - x\cos x$

Q4.

Find d^2y/dx^2: $y=\log x$

$y' = 1/x$, $y'' = -1/x^2$

Q5.

Find d^2y/dx^2: $y=x^3\log x$

Solution: $y'=3x^2\log x+x^2$. $y''=6x\log x+3x+2x$.

$y'' = 6x \log x + 5x$

Q6.

Find d^2y/dx^2: $y=e^x\sin 5x$

Solution: $y'=e^x(\sin 5x+5\cos 5x)$. $y''=e^x(\sin 5x+5\cos 5x)+e^x(5\cos 5x-25\sin 5x)$.

$y'' = e^x(10\cos 5x - 24\sin 5x)$

Q7.

Find d^2y/dx^2: $y=e^{6x}\cos 3x$

Solution: $y'=e^{6x}(6\cos 3x-3\sin 3x)$. $y''=6e^{6x}(6\cos 3x-3\sin 3x)+e^{6x}(-18\sin 3x-9\cos 3x)$.

$y'' = e^{6x}(27\cos 3x - 36\sin 3x)$

Q8.

Find d^2y/dx^2: $y=\tan^{-1}x$

Solution: $y'=1/(1+x^2)$.

$y'' = -2x/(1+x^2)^2$

Q9.

Find d^2y/dx^2: $y=\log(\log x)$

Solution: $y'=1/(x\log x)$.

$y'' = -(\log x+1) / (x \log x)^2$

Q10.

Find d^2y/dx^2: $y=\sin(\log x)$

Solution: $y'=\cos(\log x)/x$. $y''=-\sin(\log x)/x^2 - \cos(\log x)/x^2$.

$y'' = -[\sin(\log x)+\cos(\log x)] / x^2$

Q11.

If $y=5\cos x - 3\sin x$, prove that $d^2y/dx^2 + y = 0$.

Solution: $y'=-5\sin x-3\cos x$. $y''=-5\cos x+3\sin x = -(5\cos x-3\sin x) = -y$. Hence $y''+y=0$. Proved.

Q12.

If $y=\cos^{-1}x$, find d^2y/dx^2 in terms of y alone.

Solution: $y'=-(1-x^2)^{-1/2}$. $y''=-x(1-x^2)^{-3/2}$. Substituting $x=\cos y$, $1-x^2=\sin^2 y$:

$y'' = -\cos y / \sin^3 y$

Q13.

If $y=3\cos(\log x)+4\sin(\log x)$, show that $x^2y'' + xy' + y = 0$.

Solution: $y' = [1/x][4\cos(\log x)-3\sin(\log x)]$, so $xy' = 4\cos(\log x)-3\sin(\log x)$. Differentiating xy' again: $y'+xy'' = -(1/x)[4\sin(\log x)+3\cos(\log x)]$, so $xy'+x^2y'' = -[4\sin(\log x)+3\cos(\log x)] = -y$ (matching the original expression). Hence $x^2y''+xy'+y=0$. Proved.

Q14.

If $y=Ae^{mx}+Be^{nx}$, show that $y'' - (m+n)y' + mny = 0$.

Solution: $y'=Ame^{mx}+Bne^{nx}$, $y''=Am^2e^{mx}+Bn^2e^{nx}$. Then $y''-(m+n)y' = Ae^{mx}(m^2-m^2-mn)+Be^{nx}(n^2-mn-n^2) = -mn(Ae^{mx}+Be^{nx}) = -mny$. Hence $y''-(m+n)y'+mny=0$. Proved.

Q15.

If $y=500e^{7x}+600e^{-7x}$, show that $y''=49y$.

Solution: $y'=3500e^{7x}-4200e^{-7x}$. $y''=24500e^{7x}+29400e^{-7x} = 49(500e^{7x}+600e^{-7x}) = 49y$. Proved.

Q16.

If $e^y(x+1)=1$, show that $y'' = (y')^2$.

Solution: From the equation, $e^y=1/(x+1)$, so $y = -\log(x+1)$. Then $y'=-1/(x+1)$, and $y''=1/(x+1)^2 = (y')^2$. Proved.

Q17.

If $y=(\tan^{-1}x)^2$, show that $(x^2+1)^2y'' + 2x(x^2+1)y' = 2$.

Solution: $y'=2\tan^{-1}x/(1+x^2)$, so $(1+x^2)y'=2\tan^{-1}x$. Differentiating: $2xy'+(1+x^2)y''=2/(1+x^2)$. Multiplying through by $(1+x^2)$: $2x(1+x^2)y'+(1+x^2)^2y''=2$. Proved.

Exercise 5.8 – Rolle's Theorem and Mean Value Theorem (Q1–6)

Q1.

Verify Rolle's theorem for $f(x)=x^2+2x-8$, $x\in[-4,2]$.

Solution: f is a polynomial, so continuous on $[-4,2]$ and differentiable on $(-4,2)$. $f(-4)=16-8-8=0$, $f(2)=4+4-8=0$, so $f(-4)=f(2)$. Rolle's theorem applies. $f'(x)=2x+2=0 \Rightarrow x=-1$.

$c = -1 \in (-4,2)$. Verified.

Q2.

Examine whether Rolle's theorem is applicable: (i) $f(x)=[x]$, $x\in[5,9]$ (ii) $f(x)=[x]$, $x\in[-2,2]$ (iii) $f(x)=x^2-1$, $x\in[1,2]$.

Solution: (i), (ii): The greatest integer function $[x]$ is discontinuous at every integer in these intervals, so f fails to be continuous on the closed interval — Rolle's theorem is not applicable.
(iii): f is continuous and differentiable everywhere, but $f(1)=0$ and $f(2)=3$, so $f(1)\neq f(2)$ — the endpoint condition fails, so Rolle's theorem is not applicable (though f is otherwise well-behaved).

Q3.

If $f:[-5,5]\to R$ is differentiable and $f'(x)$ does not vanish anywhere, prove that $f(-5)\neq f(5)$.

Solution: Suppose, for contradiction, that $f(-5)=f(5)$. Since f is differentiable (hence continuous) on $[-5,5]$, Rolle's theorem would guarantee some $c\in(-5,5)$ with $f'(c)=0$. This contradicts the hypothesis that f' never vanishes. Hence the assumption is false, and $f(-5)\neq f(5)$.

Q4.

Verify the Mean Value Theorem for $f(x)=x^2-4x-3$ on $[1,4]$.

Solution: f is continuous and differentiable (polynomial). $f(1)=1-4-3=-6$, $f(4)=16-16-3=-3$. $[f(4)-f(1)]/(4-1) = 3/3 = 1$. $f'(x)=2x-4$; setting $2c-4=1$ gives $c=5/2$.

$c = 5/2 \in (1,4)$. Verified.

Q5.

Verify the Mean Value Theorem for $f(x)=x^3-5x^2-3x$ on $[1,3]$. Find all $c\in(1,3)$ for which $f'(c)$ equals the mean slope.

Solution: $f(1)=1-5-3=-7$, $f(3)=27-45-9=-27$. $[f(3)-f(1)]/(3-1) = -20/2 = -10$. $f'(x)=3x^2-10x-3$. Setting $3x^2-10x-3=-10$: $3x^2-10x+7=0 \Rightarrow x=(10\pm4)/6 = 7/3$ or 1.

Only $c = 7/3$ lies in the open interval $(1,3)$. Verified.

Q6.

Examine the applicability of the Mean Value Theorem for: (i) $f(x)=[x]$, $x\in[5,9]$ (ii) $f(x)=[x]$, $x\in[-2,2]$ (iii) $f(x)=x^2-1$, $x\in[1,2]$.

Solution: (i), (ii): $[x]$ is discontinuous at integers within these intervals — MVT is not applicable.
(iii): f is continuous and differentiable on $[1,2]$. $[f(2)-f(1)]/(2-1) = (3-0)/1 = 3$. $f'(x)=2x=3 \Rightarrow c=3/2 \in (1,2)$. MVT holds.

End of Exercises 5.1–5.8. The Miscellaneous Exercise for Chapter 5, and the next chapter, will follow as separate parts of this series.