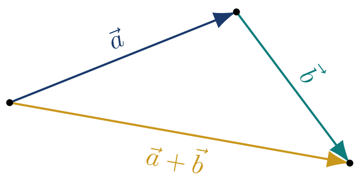
NCERT CLASS 12 MATHEMATICS

Chapter-wise Solutions Series

CHAPTER 10

VECTOR ALGEBRA

Complete Step-by-Step NCERT Solutions



What's Inside

- Concept summary with diagrams before every exercise
- Complete, verified solutions – Exercises 10.1 to 10.4
- Miscellaneous Exercise fully solved
- Clean step-by-step working for every problem
- Quick formula sheet for last-minute revision

Prepared for Class 12 CBSE / NCERT Mathematics students

Suitable for board exam preparation and quick revision

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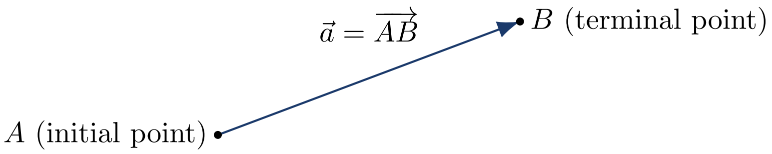
How to use this booklet: Each exercise opens with a short concept box recapping the formulas (and a diagram, where helpful) you need. Every question is shown in a blue box, immediately followed by its complete worked solution in a green box. Work through the problems yourself first, then check your steps against the solution.

Chapter Overview

A **vector** is a quantity that has both magnitude and direction (e.g. displacement, velocity, force), unlike a **scalar**, which has only magnitude (e.g. mass, time, distance). A vector $\overrightarrow{AB}$ starts at the *initial point* A and ends at the *terminal point* B.



Key Concepts & Formulas

1. Position vector: of a point $P(x, y, z)$ is $\overrightarrow{OP} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$, with magnitude $|\overrightarrow{OP}| = \sqrt{x^2 + y^2 + z^2}$.
2. Direction cosines (l, m, n) of a vector are the cosines of the angles it makes with the x, y, z axes; $l^2 + m^2 + n^2 = 1$. Direction ratios are any numbers proportional to the direction cosines.
3. Types of vectors: zero vector $\vec{0}$; unit vector ($|\hat{a}| = 1$, so $\hat{a} = \vec{a}/|\vec{a}|$); coinitial vectors (same initial point); collinear vectors (parallel to the same line); equal vectors (same magnitude & direction); negative of a vector ($-\vec{a}$ has same magnitude, opposite direction).
4. Triangle law of addition: $\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$.
5. Section formula: position vector of R dividing PQ (with p.v. $\vec{p}, \vec{q}$) internally in ratio $m : n$ is $\dfrac{m\vec{q} + n\vec{p}}{m + n}$; externally, $\dfrac{m\vec{q} - n\vec{p}}{m - n}$.
6. Scalar (dot) product: $\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos\theta$; gives a scalar. $\vec{a} \perp \vec{b} \iff \vec{a} \cdot \vec{b} = 0$. Projection of $\vec{a}$ on $\vec{b} = \dfrac{\vec{a} \cdot \vec{b}}{|\vec{b}|}$.
7. Vector (cross) product: $\vec{a} \times \vec{b} = |\vec{a}||\vec{b}| \sin\theta\, \hat{n}$; gives a vector perpendicular to both $\vec{a}, \vec{b}$. $\vec{a} \parallel \vec{b} \iff \vec{a} \times \vec{b} = \vec{0}$. Area of triangle with sides $\vec{a}, \vec{b} = \frac{1}{2}|\vec{a} \times \vec{b}|$; area of parallelogram $= |\vec{a} \times \vec{b}|$.

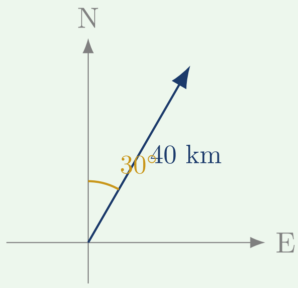
Exercise 10.1 | Basic Concepts and Types of Vectors

Key Concepts & Formulas

A scalar has only magnitude; a vector has both magnitude and direction. Vectors with the same magnitude and direction are equal, regardless of their initial point. Collinear vectors lie along the same or parallel lines (they need not have equal magnitude).

Question 1. Represent graphically a displacement of 40 km, $30°$ east of north.

Solution: The displacement vector has magnitude 40 km, and its direction is measured $30°$ from the north direction, rotated towards the east:



The arrow shows a vector of length 40 (representing km) making a $30°$ angle with the north direction, swung towards the east.

Question 2. Classify the following measures as scalars and vectors:
(i) 10 kg (ii) 2 metres north-west (iii) $40°$ (iv) 40 watts (v) 10^{-19} coulomb (vi) 20 m/s^2

Solution:

Quantity	Type
(i) 10 kg (mass)	Scalar
(ii) 2 m north-west (displacement)	Vector
(iii) $40°$ (temperature)	Scalar
(iv) 40 watts (power)	Scalar
(v) 10^{-19} coulomb (electric charge)	Scalar
(vi) 20 m/s^2 (acceleration)	Vector

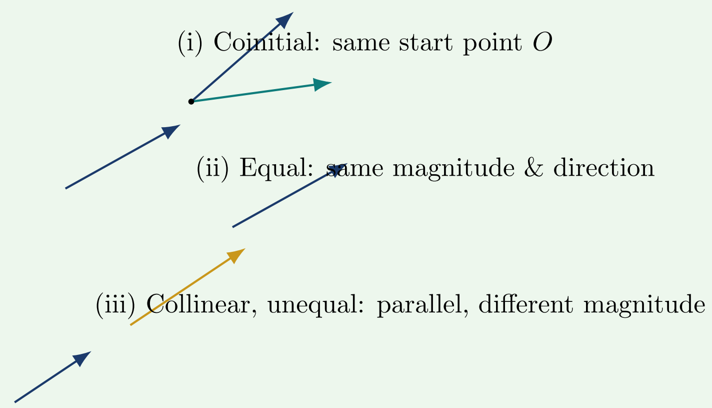
Question 3. Classify the following as scalar and vector quantities:
(i) time period (ii) distance (iii) force (iv) velocity (v) work done

Solution:

Quantity	Type
(i) Time period	Scalar
(ii) Distance	Scalar
(iii) Force	Vector
(iv) Velocity	Vector
(v) Work done	Scalar

Question 4. Explain, with the help of a diagram, what is meant by (i) coinitial vectors (ii) equal vectors (iii) collinear but not equal vectors.

Solution:



(i) Coinitial vectors have the same starting point.
(ii) Equal vectors have equal magnitude *and* the same direction (they may have different initial points).
(iii) Collinear but not equal vectors lie along the same or parallel lines but differ in magnitude and/or direction (they are not equal).

Question 5. Answer the following as true or false:
(i) $\vec{a}$ and $-\vec{a}$ are collinear.
(ii) Two collinear vectors are always equal in magnitude.
(iii) Two vectors having the same magnitude are collinear.
(iv) Two collinear vectors having the same magnitude are equal.

Solution: (i) True. $-\vec{a}$ is parallel to $\vec{a}$ (just opposite in direction), so both lie along the same line.
(ii) False. Collinear only means the vectors are parallel to the same line; their magnitudes can be different.
(iii) False. Equal magnitude does not force the vectors to be parallel – they could point in completely different directions.
(iv) False. Two collinear vectors of the same magnitude could point in *opposite* directions (e.g. $\vec{a}$ and $-\vec{a}$), in which case they are not equal.

Exercise 10.2 | Algebra of Vectors

Key Concepts & Formulas

For $\vec{a} = a_1\hat{\imath} + a_2\hat{\jmath} + a_3\hat{k}$: magnitude $|\vec{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$, unit vector $\hat{a} = \vec{a}/|\vec{a}|$. If A, B have position vectors $\vec{p}, \vec{q}$, then $\overrightarrow{AB} = \vec{q} - \vec{p}$. Section formula (internal): $\vec{r} = \dfrac{m\vec{q} + n\vec{p}}{m+n}$; (external): $\vec{r} = \dfrac{m\vec{q} - n\vec{p}}{m-n}$.

Question 1. Find the unit vector in the direction of the vector $\vec{a} = \hat{\imath} + \hat{\jmath} + 2\hat{k}$.

Solution:

$$|\vec{a}| = \sqrt{1^2 + 1^2 + 2^2} = \sqrt{6}$$
$$\hat{a} = \frac{\vec{a}}{|\vec{a}|} = \boxed{\frac{1}{\sqrt{6}}\hat{\imath} + \frac{1}{\sqrt{6}}\hat{\jmath} + \frac{2}{\sqrt{6}}\hat{k}}$$

Question 2. Find the unit vector in the direction of vector $\overrightarrow{PQ}$, where $P(1, 2, 3)$ and $Q(4, 5, 6)$.

Solution:

$$\overrightarrow{PQ} = (4-1)\hat{\imath} + (5-2)\hat{\jmath} + (6-3)\hat{k} = 3\hat{\imath} + 3\hat{\jmath} + 3\hat{k}$$
$$|\overrightarrow{PQ}| = \sqrt{9+9+9} = \sqrt{27} = 3\sqrt{3}$$
$$\widehat{PQ} = \frac{3\hat{\imath} + 3\hat{\jmath} + 3\hat{k}}{3\sqrt{3}} = \boxed{\frac{1}{\sqrt{3}}\hat{\imath} + \frac{1}{\sqrt{3}}\hat{\jmath} + \frac{1}{\sqrt{3}}\hat{k}}$$

Question 3. Find the sum of the vectors $\vec{a} = \hat{\imath} - 2\hat{\jmath} + \hat{k}$, $\vec{b} = -2\hat{\imath} + 4\hat{\jmath} + 5\hat{k}$, $\vec{c} = \hat{\imath} - 6\hat{\jmath} - 7\hat{k}$.

Solution:

$$\vec{a} + \vec{b} + \vec{c} = (1-2+1)\hat{\imath} + (-2+4-6)\hat{\jmath} + (1+5-7)\hat{k}$$
$$= 0\hat{\imath} - 4\hat{\jmath} - 1\hat{k} = \boxed{-4\hat{\jmath} - \hat{k}}$$

Question 4. Find the unit vector in the direction of $\vec{a}+\vec{b}$, where $\vec{a} = \hat{\imath}+\hat{\jmath}+2\hat{k}$ and $\vec{b} = 2\hat{\imath}+\hat{\jmath}-2\hat{k}$.

Solution:

$$\vec{a} + \vec{b} = 3\hat{\imath} + 2\hat{\jmath} + 0\hat{k}$$
$$|\vec{a} + \vec{b}| = \sqrt{9+4} = \sqrt{13}$$
$$\widehat{(a+b)} = \boxed{\frac{3}{\sqrt{13}}\hat{\imath} + \frac{2}{\sqrt{13}}\hat{\jmath}}$$

Question 5. Find a vector in the direction of vector $\vec{a} = \hat{\imath} - 2\hat{\jmath}$ that has magnitude 7 units.

Solution:

$$|\vec{a}| = \sqrt{1+4} = \sqrt{5}, \qquad \hat{a} = \frac{1}{\sqrt{5}}(\hat{\imath} - 2\hat{\jmath})$$

Required vector $= 7\hat{a}$:

$$\boxed{\frac{7}{\sqrt{5}}\hat{\imath} - \frac{14}{\sqrt{5}}\hat{\jmath}}$$

Question 6. Show that the vectors $2\hat{\imath} - 3\hat{\jmath} + 4\hat{k}$ and $-4\hat{\imath} + 6\hat{\jmath} - 8\hat{k}$ are collinear.

Solution: Let $\vec{a} = 2\hat{\imath} - 3\hat{\jmath} + 4\hat{k}$ and $\vec{b} = -4\hat{\imath} + 6\hat{\jmath} - 8\hat{k}$. Observe

$$\vec{b} = -2(2\hat{\imath} - 3\hat{\jmath} + 4\hat{k}) = -2\vec{a}$$

Since $\vec{b}$ is a scalar multiple of $\vec{a}$, the two vectors are parallel, i.e. $\boxed{\text{collinear}}$.

Question 7. Find the direction cosines of the vector $\hat{\imath} + 2\hat{\jmath} + 3\hat{k}$.

Solution:

$$|\vec{v}| = \sqrt{1^2 + 2^2 + 3^2} = \sqrt{14}$$

$$\boxed{(l, m, n) = \left(\frac{1}{\sqrt{14}}, \frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}}\right)}$$

Question 8. Find the direction cosines of the vector joining the points $A(1, 2, -3)$ and $B(-1, -2, 1)$, directed from A to B.

Solution:

$$\overrightarrow{AB} = (-1-1)\hat{\imath} + (-2-2)\hat{\jmath} + (1-(-3))\hat{k} = -2\hat{\imath} - 4\hat{\jmath} + 4\hat{k}$$

$$|\overrightarrow{AB}| = \sqrt{4 + 16 + 16} = \sqrt{36} = 6$$

$$\boxed{(l, m, n) = \left(-\frac{1}{3}, -\frac{2}{3}, \frac{2}{3}\right)}$$

Question 9. Show that the vector $\hat{\imath} + \hat{\jmath} + \hat{k}$ is equally inclined to the axes OX, OY, OZ.

Solution:

$$|\vec{v}| = \sqrt{1+1+1} = \sqrt{3}, \qquad (l, m, n) = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$$

Since $l = m = n$, the angles made with all three axes are equal, so the vector is $\boxed{\text{equally inclined to } OX, OY, OZ}$.

Question 10. Find the position vector of a point R which divides the line joining two points P and Q, whose position vectors are $\hat{\imath} + 2\hat{\jmath} - \hat{k}$ and $-\hat{\imath} + \hat{\jmath} + \hat{k}$ respectively, in the ratio $2 : 1$ (i) internally (ii) externally.

Solution: Let $\vec{p} = \hat{\imath} + 2\hat{\jmath} - \hat{k}$, $\vec{q} = -\hat{\imath} + \hat{\jmath} + \hat{k}$, ratio $m : n = 2 : 1$.
(i) Internal division:

$$\vec{r} = \frac{m\vec{q} + n\vec{p}}{m+n} = \frac{2(-\hat{\imath} + \hat{\jmath} + \hat{k}) + 1(\hat{\imath} + 2\hat{\jmath} - \hat{k})}{3}$$

$$= \frac{-2\hat{\imath} + 2\hat{\jmath} + 2\hat{k} + \hat{\imath} + 2\hat{\jmath} - \hat{k}}{3} = \frac{-\hat{\imath} + 4\hat{\jmath} + \hat{k}}{3}$$

$$\boxed{\vec{r} = -\tfrac{1}{3}\hat{\imath} + \tfrac{4}{3}\hat{\jmath} + \tfrac{1}{3}\hat{k}}$$

(ii) External division:

$$\vec{r} = \frac{m\vec{q} - n\vec{p}}{m-n} = \frac{2(-\hat{\imath} + \hat{\jmath} + \hat{k}) - 1(\hat{\imath} + 2\hat{\jmath} - \hat{k})}{1}$$

$$= -2\hat{\imath} + 2\hat{\jmath} + 2\hat{k} - \hat{\imath} - 2\hat{\jmath} + \hat{k} = -3\hat{\imath} + 0\hat{\jmath} + 3\hat{k}$$

$$\boxed{\vec{r} = -3\hat{\imath} + 3\hat{k}}$$

Question 11. Find the position vector of the midpoint of the line segment joining the points $P(2, 3, 4)$ and $Q(4, 1, -2)$.

Solution: Midpoint $= \left(\frac{2+4}{2}, \frac{3+1}{2}, \frac{4+(-2)}{2}\right) = (3, 2, 1)$

$$\boxed{\overrightarrow{OM} = 3\hat{\imath} + 2\hat{\jmath} + \hat{k}}$$

Question 12. Show that the points A, B, C with position vectors $\vec{a} = 3\hat{\imath} - 4\hat{\jmath} - 4\hat{k}$, $\vec{b} = 2\hat{\imath} - \hat{\jmath} + \hat{k}$, $\vec{c} = \hat{\imath} - 3\hat{\jmath} - 5\hat{k}$ respectively form the vertices of a right-angled triangle.

Solution:

$$\begin{aligned}
\overrightarrow{AB} &= \vec{b} - \vec{a} = -\hat{\imath} + 3\hat{\jmath} + 5\hat{k}, & |\overrightarrow{AB}|^2 &= 1 + 9 + 25 = 35\\
\overrightarrow{BC} &= \vec{c} - \vec{b} = -\hat{\imath} - 2\hat{\jmath} - 6\hat{k}, & |\overrightarrow{BC}|^2 &= 1 + 4 + 36 = 41\\
\overrightarrow{CA} &= \vec{a} - \vec{c} = 2\hat{\imath} - \hat{\jmath} + \hat{k}, & |\overrightarrow{CA}|^2 &= 4 + 1 + 1 = 6
\end{aligned}$$

Since $|\overrightarrow{AB}|^2 + |\overrightarrow{CA}|^2 = 35 + 6 = 41 = |\overrightarrow{BC}|^2$, by the converse of the Pythagoras theorem, $\triangle ABC$ is $\boxed{\text{right-angled at } A}$.

Exercise 10.3 | Scalar (Dot) Product of Vectors

Key Concepts & Formulas

$\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta = a_1b_1 + a_2b_2 + a_3b_3$. Perpendicular vectors satisfy $\vec{a}\cdot\vec{b} = 0$. Projection of $\vec{a}$ on $\vec{b}$ is $\dfrac{\vec{a}\cdot\vec{b}}{|\vec{b}|}$. Also $\vec{a}\cdot\vec{a} = |\vec{a}|^2$.

Question 1. Find the angle between two vectors $\vec{a}$ and $\vec{b}$ with magnitudes $\sqrt{3}$ and 2 respectively, given $\vec{a}\cdot\vec{b} = \sqrt{6}$.

Solution:

$$\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|} = \frac{\sqrt{6}}{\sqrt{3}\cdot 2} = \frac{\sqrt{6}}{2\sqrt{3}} = \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}$$

$$\boxed{\theta = \frac{\pi}{4}}$$

Question 2. Find the angle between the vectors $\vec{a} = \hat{\imath} - \hat{\jmath} + \hat{k}$ and $\vec{b} = \hat{\imath} + \hat{\jmath} - \hat{k}$.

Solution:

$$\vec{a}\cdot\vec{b} = (1)(1) + (-1)(1) + (1)(-1) = 1 - 1 - 1 = -1$$

$$|\vec{a}| = \sqrt{1+1+1} = \sqrt{3}, \qquad |\vec{b}| = \sqrt{1+1+1} = \sqrt{3}$$

$$\cos\theta = \frac{-1}{\sqrt{3}\cdot\sqrt{3}} = -\frac{1}{3}$$

$$\boxed{\theta = \cos^{-1}\left(-\frac{1}{3}\right)}$$

Question 3. Find the projection of the vector $\vec{a} = 2\hat{\imath} + 3\hat{\jmath} + 2\hat{k}$ on the vector $\vec{b} = \hat{\imath} + 2\hat{\jmath} + \hat{k}$.

Solution:

$$\vec{a}\cdot\vec{b} = (2)(1) + (3)(2) + (2)(1) = 2 + 6 + 2 = 10, \qquad |\vec{b}| = \sqrt{1+4+1} = \sqrt{6}$$

$$\text{Projection} = \frac{\vec{a}\cdot\vec{b}}{|\vec{b}|} = \boxed{\frac{10}{\sqrt{6}}}$$

Question 4. Find the projection of the vector $\hat{\imath} - \hat{\jmath}$ on the vector $\hat{\imath} + \hat{\jmath}$.

Solution:

$$(\hat{\imath} - \hat{\jmath})\cdot(\hat{\imath} + \hat{\jmath}) = (1)(1) + (-1)(1) = 0$$

$$\text{Projection} = \frac{0}{|\hat{\imath} + \hat{\jmath}|} = \boxed{0}$$

(The two vectors are perpendicular.)

Question 5. Find λ such that the vectors $\vec{a} = 2\hat{\imath} + \lambda\hat{\jmath} + \hat{k}$ and $\vec{b} = \hat{\imath} + 2\hat{\jmath} + 3\hat{k}$ are perpendicular.

Solution: Perpendicular $\implies \vec{a} \cdot \vec{b} = 0$:

$$2(1) + \lambda(2) + 1(3) = 0 \implies 2 + 2\lambda + 3 = 0 \implies 2\lambda = -5$$

$$\boxed{\lambda = -\frac{5}{2}}$$

Question 6. If $\vec{a}$ is a unit vector and $(\vec{x} - \vec{a}) \cdot (\vec{x} + \vec{a}) = 8$, find $|\vec{x}|$.

Solution:

$$(\vec{x} - \vec{a}) \cdot (\vec{x} + \vec{a}) = \vec{x} \cdot \vec{x} - \vec{a} \cdot \vec{a} = |\vec{x}|^2 - |\vec{a}|^2$$

Since $\vec{a}$ is a unit vector, $|\vec{a}|^2 = 1$:

$$|\vec{x}|^2 - 1 = 8 \implies |\vec{x}|^2 = 9$$

$$\boxed{|\vec{x}| = 3}$$

Question 7. If $\vec{a}, \vec{b}, \vec{c}$ are unit vectors such that $\vec{a} + \vec{b} + \vec{c} = \vec{0}$, find the value of $\vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{c} + \vec{c} \cdot \vec{a}$.

Solution: Since $\vec{a} + \vec{b} + \vec{c} = \vec{0}$, we have $|\vec{a} + \vec{b} + \vec{c}|^2 = 0$:

$$|\vec{a}|^2 + |\vec{b}|^2 + |\vec{c}|^2 + 2(\vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{c} + \vec{c} \cdot \vec{a}) = 0$$

As $\vec{a}, \vec{b}, \vec{c}$ are unit vectors, $|\vec{a}|^2 = |\vec{b}|^2 = |\vec{c}|^2 = 1$:

$$1 + 1 + 1 + 2(\vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{c} + \vec{c} \cdot \vec{a}) = 0$$

$$\boxed{\vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{c} + \vec{c} \cdot \vec{a} = -\frac{3}{2}}$$

Question 8. Show that $|\vec{a}|\vec{b} + |\vec{b}|\vec{a}$ is perpendicular to $|\vec{a}|\vec{b} - |\vec{b}|\vec{a}$, for any two nonzero vectors $\vec{a}$ and $\vec{b}$.

Solution: Let $\vec{u} = |\vec{a}|\vec{b} + |\vec{b}|\vec{a}$ and $\vec{v} = |\vec{a}|\vec{b} - |\vec{b}|\vec{a}$.

$$\begin{aligned}
\vec{u} \cdot \vec{v} &= \big(|\vec{a}|\vec{b} + |\vec{b}|\vec{a}\big) \cdot \big(|\vec{a}|\vec{b} - |\vec{b}|\vec{a}\big) \\
&= |\vec{a}|^2(\vec{b} \cdot \vec{b}) - |\vec{a}||\vec{b}|(\vec{b} \cdot \vec{a}) + |\vec{a}||\vec{b}|(\vec{a} \cdot \vec{b}) - |\vec{b}|^2(\vec{a} \cdot \vec{a}) \\
&= |\vec{a}|^2|\vec{b}|^2 - |\vec{a}||\vec{b}|(\vec{a} \cdot \vec{b}) + |\vec{a}||\vec{b}|(\vec{a} \cdot \vec{b}) - |\vec{b}|^2|\vec{a}|^2 \\
&= |\vec{a}|^2|\vec{b}|^2 - |\vec{b}|^2|\vec{a}|^2 = 0
\end{aligned}$$

Since $\vec{u} \cdot \vec{v} = 0$, $\boxed{\vec{u} \perp \vec{v}}$.

Exercise 10.4 | Vector (Cross) Product of Vectors

Key Concepts & Formulas

$\vec{a}\times\vec{b} = |\vec{a}||\vec{b}|\sin\theta\,\hat{n} = \begin{vmatrix}\hat{\imath} & \hat{\jmath} & \hat{k}\\ a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\end{vmatrix}$. Area of $\triangle = \frac{1}{2}|\overrightarrow{AB}\times\overrightarrow{AC}|$. Parallel vectors satisfy $\vec{a}\times\vec{b} = \vec{0}$.

Question 1. Find $|\vec{a}\times\vec{b}|$ if $\vec{a} = \hat{\imath} - 7\hat{\jmath} + 7\hat{k}$ and $\vec{b} = 3\hat{\imath} - 2\hat{\jmath} + 2\hat{k}$.

Solution:

$$\vec{a}\times\vec{b} = \begin{vmatrix}\hat{\imath} & \hat{\jmath} & \hat{k}\\ 1 & -7 & 7\\ 3 & -2 & 2\end{vmatrix}$$

$$\begin{aligned}&= \hat{\imath}\big[(-7)(2) - (7)(-2)\big] - \hat{\jmath}\big[(1)(2) - (7)(3)\big] + \hat{k}\big[(1)(-2) - (-7)(3)\big]\\ &= \hat{\imath}(-14 + 14) - \hat{\jmath}(2 - 21) + \hat{k}(-2 + 21)\\ &= 0\hat{\imath} + 19\hat{\jmath} + 19\hat{k}\end{aligned}$$

$$|\vec{a}\times\vec{b}| = \sqrt{0^2 + 19^2 + 19^2} = \sqrt{722} = \boxed{19\sqrt{2}}$$

Question 2. Find a unit vector perpendicular to each of the vectors $\vec{a}+\vec{b}$ and $\vec{a}-\vec{b}$, where $\vec{a} = \hat{\imath} + \hat{\jmath} + \hat{k}$, $\vec{b} = \hat{\imath} + 2\hat{\jmath} + 3\hat{k}$.

Solution:

$$\vec{a}+\vec{b} = 2\hat{\imath} + 3\hat{\jmath} + 4\hat{k}, \qquad \vec{a}-\vec{b} = 0\hat{\imath} - \hat{\jmath} - 2\hat{k}$$

$$(\vec{a}+\vec{b})\times(\vec{a}-\vec{b}) = \begin{vmatrix}\hat{\imath} & \hat{\jmath} & \hat{k}\\ 2 & 3 & 4\\ 0 & -1 & -2\end{vmatrix}$$

$$\begin{aligned}&= \hat{\imath}\big[(3)(-2) - (4)(-1)\big] - \hat{\jmath}\big[(2)(-2) - (4)(0)\big] + \hat{k}\big[(2)(-1) - (3)(0)\big]\\ &= \hat{\imath}(-6 + 4) - \hat{\jmath}(-4 - 0) + \hat{k}(-2 - 0) = -2\hat{\imath} + 4\hat{\jmath} - 2\hat{k}\end{aligned}$$

$$\text{magnitude} = \sqrt{4 + 16 + 4} = \sqrt{24} = 2\sqrt{6}$$

$$\boxed{\hat{n} = \frac{-1}{\sqrt{6}}\hat{\imath} + \frac{2}{\sqrt{6}}\hat{\jmath} - \frac{1}{\sqrt{6}}\hat{k}}$$

Question 3. If a unit vector $\vec{a}$ makes angles $\pi/3$ with $\hat{\imath}$, $\pi/4$ with $\hat{\jmath}$ and an acute angle θ with $\hat{k}$, find θ and hence the components of $\vec{a}$.

Solution: Since $\vec{a}$ is a unit vector, its direction cosines satisfy $l^2 + m^2 + n^2 = 1$:

$$\cos^2\frac{\pi}{3} + \cos^2\frac{\pi}{4} + \cos^2\theta = 1$$

$$\left(\frac{1}{2}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2 + \cos^2\theta = 1$$

$$\frac{1}{4} + \frac{1}{2} + \cos^2\theta = 1 \implies \cos^2\theta = \frac{1}{4} \implies \cos\theta = \pm\frac{1}{2}$$

Since θ is acute, $\cos\theta = \frac{1}{2}$, so $\boxed{\theta = \frac{\pi}{3}}$.

$$\vec{a} = \cos\frac{\pi}{3}\,\hat{\imath} + \cos\frac{\pi}{4}\,\hat{\jmath} + \cos\frac{\pi}{3}\,\hat{k} = \boxed{\frac{1}{2}\hat{\imath} + \frac{1}{\sqrt{2}}\hat{\jmath} + \frac{1}{2}\hat{k}}$$

Question 4. Show that $(\vec{a} - \vec{b}) \times (\vec{a} + \vec{b}) = 2(\vec{a} \times \vec{b})$.

Solution: Expanding using distributivity of the cross product:

$$\begin{aligned}(\vec{a} - \vec{b}) \times (\vec{a} + \vec{b}) &= \vec{a} \times \vec{a} + \vec{a} \times \vec{b} - \vec{b} \times \vec{a} - \vec{b} \times \vec{b} \\ &= \vec{0} + \vec{a} \times \vec{b} - (-\vec{a} \times \vec{b}) - \vec{0} \qquad (\text{since } \vec{b} \times \vec{a} = -\vec{a} \times \vec{b}) \\ &= \vec{a} \times \vec{b} + \vec{a} \times \vec{b} \\ &= \boxed{2(\vec{a} \times \vec{b})}\end{aligned}$$

Question 5. Find λ and μ if $(2\hat{\imath} + 6\hat{\jmath} + 27\hat{k}) \times (\hat{\imath} + \lambda\hat{\jmath} + \mu\hat{k}) = \vec{0}$.

Solution: The cross product is zero exactly when the vectors are parallel, i.e. their components are proportional:

$$\frac{2}{1} = \frac{6}{\lambda} = \frac{27}{\mu}$$

From $\frac{2}{1} = 2$: $\frac{6}{\lambda} = 2 \implies \lambda = 3$, and $\frac{27}{\mu} = 2 \implies \mu = \frac{27}{2}$.

$$\boxed{\lambda = 3, \quad \mu = \frac{27}{2}}$$

Question 6. If $|\vec{a}| = 3$, $|\vec{b}| = \sqrt{\frac{2}{3}}$, and $\vec{a} \times \vec{b}$ is a unit vector, find the angle between $\vec{a}$ and $\vec{b}$.

Solution:

$$|\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}|\sin\theta = 1$$

$$3 \cdot \sqrt{\frac{2}{3}} \cdot \sin\theta = 1 \implies \sqrt{6}\sin\theta = 1$$

$$\boxed{\theta = \sin^{-1}\left(\frac{1}{\sqrt{6}}\right)}$$

Question 7. Let vectors $\vec{a}, \vec{b}, \vec{c}$ be such that $|\vec{a}| = 3, |\vec{b}| = 4, |\vec{c}| = 5$ and each one is perpendicular to the sum of the other two. Find $|\vec{a} + \vec{b} + \vec{c}|$.

Solution: We are given:

$$\vec{a} \cdot (\vec{b} + \vec{c}) = 0, \qquad \vec{b} \cdot (\vec{a} + \vec{c}) = 0, \qquad \vec{c} \cdot (\vec{a} + \vec{b}) = 0$$

Expanding and adding all three:

$$2(\vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{c} + \vec{c} \cdot \vec{a}) = 0 \implies \vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{c} + \vec{c} \cdot \vec{a} = 0$$

Now,

$$\begin{aligned} |\vec{a} + \vec{b} + \vec{c}|^2 &= |\vec{a}|^2 + |\vec{b}|^2 + |\vec{c}|^2 + 2(\vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{c} + \vec{c} \cdot \vec{a}) \\ &= 9 + 16 + 25 + 2(0) = 50 \end{aligned}$$

$$\boxed{|\vec{a} + \vec{b} + \vec{c}| = 5\sqrt{2}}$$

Question 8. Show that the area of the parallelogram whose diagonals are $\vec{a}$ and $\vec{b}$ is $\dfrac{1}{2}|\vec{a} \times \vec{b}|$.

Solution: Let the parallelogram have adjacent sides $\vec{p}, \vec{q}$, so the diagonals are $\vec{a} = \vec{p} + \vec{q}$ and $\vec{b} = \vec{p} - \vec{q}$. Then

$$\begin{aligned} \vec{a} \times \vec{b} &= (\vec{p} + \vec{q}) \times (\vec{p} - \vec{q}) \\ &= \vec{p} \times \vec{p} - \vec{p} \times \vec{q} + \vec{q} \times \vec{p} - \vec{q} \times \vec{q} \\ &= \vec{0} - \vec{p} \times \vec{q} - \vec{p} \times \vec{q} - \vec{0} = -2(\vec{p} \times \vec{q}) \end{aligned}$$

Taking magnitudes: $|\vec{a} \times \vec{b}| = 2|\vec{p} \times \vec{q}|$. Since the area of the parallelogram with sides $\vec{p}, \vec{q}$ is $|\vec{p} \times \vec{q}|$,

$$\text{Area} = |\vec{p} \times \vec{q}| = \boxed{\tfrac{1}{2}|\vec{a} \times \vec{b}|}$$

Miscellaneous Exercise

Question 1. If $\vec{a} = \hat{\imath} + \hat{\jmath} + \hat{k}$, $\vec{b} = 2\hat{\imath} - \hat{\jmath} + 3\hat{k}$, $\vec{c} = \hat{\imath} - 2\hat{\jmath} + \hat{k}$, find a unit vector parallel to $2\vec{a} - \vec{b} + 3\vec{c}$.

Solution:

$$2\vec{a} = 2\hat{\imath} + 2\hat{\jmath} + 2\hat{k}$$
$$-\vec{b} = -2\hat{\imath} + \hat{\jmath} - 3\hat{k}$$
$$3\vec{c} = 3\hat{\imath} - 6\hat{\jmath} + 3\hat{k}$$

$$2\vec{a} - \vec{b} + 3\vec{c} = (2 - 2 + 3)\hat{\imath} + (2 + 1 - 6)\hat{\jmath} + (2 - 3 + 3)\hat{k} = 3\hat{\imath} - 3\hat{\jmath} + 2\hat{k}$$
$$|2\vec{a} - \vec{b} + 3\vec{c}| = \sqrt{9 + 9 + 4} = \sqrt{22}$$
$$\boxed{\hat{n} = \frac{3}{\sqrt{22}}\hat{\imath} - \frac{3}{\sqrt{22}}\hat{\jmath} + \frac{2}{\sqrt{22}}\hat{k}}$$

Question 2. Find a unit vector perpendicular to the plane of triangle ABC, where $A(1, 1, 2)$, $B(2, 3, 5)$, $C(1, 5, 5)$.

Solution:

$$\overrightarrow{AB} = \hat{\imath} + 2\hat{\jmath} + 3\hat{k}, \qquad \overrightarrow{AC} = 0\hat{\imath} + 4\hat{\jmath} + 3\hat{k}$$
$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 2 & 3 \\ 0 & 4 & 3 \end{vmatrix}$$

$$= \hat{\imath}(2 \cdot 3 - 3 \cdot 4) - \hat{\jmath}(1 \cdot 3 - 3 \cdot 0) + \hat{k}(1 \cdot 4 - 2 \cdot 0)$$
$$= \hat{\imath}(6 - 12) - \hat{\jmath}(3 - 0) + \hat{k}(4 - 0) = -6\hat{\imath} - 3\hat{\jmath} + 4\hat{k}$$

$$|\overrightarrow{AB} \times \overrightarrow{AC}| = \sqrt{36 + 9 + 16} = \sqrt{61}$$
$$\boxed{\hat{n} = \frac{-6}{\sqrt{61}}\hat{\imath} - \frac{3}{\sqrt{61}}\hat{\jmath} + \frac{4}{\sqrt{61}}\hat{k}}$$

Question 3. Find the area of the triangle with vertices $A(1, 1, 2)$, $B(2, 3, 5)$, $C(1, 5, 5)$ (same triangle as above).

Solution: Using $\overrightarrow{AB} \times \overrightarrow{AC} = -6\hat{\imath} - 3\hat{\jmath} + 4\hat{k}$ from the previous question:

$$\text{Area} = \frac{1}{2}|\overrightarrow{AB} \times \overrightarrow{AC}| = \frac{1}{2}\sqrt{61}$$

$$\boxed{\text{Area} = \frac{\sqrt{61}}{2} \text{ sq units}}$$

Question 4. Show that the points $A(1,-2,-8)$, $B(5,0,-2)$, $C(11,3,7)$ are collinear, and find the ratio in which B divides AC.

Solution:

$$\overrightarrow{AB} = 4\hat{\imath} + 2\hat{\jmath} + 6\hat{k}, \qquad \overrightarrow{BC} = 6\hat{\imath} + 3\hat{\jmath} + 9\hat{k}$$

Observe that $\overrightarrow{BC} = \frac{3}{2}\overrightarrow{AB}$, so $\overrightarrow{AB}$ and $\overrightarrow{BC}$ are parallel and share the common point B. Hence $\boxed{A, B, C \text{ are collinear}}$.

To find the ratio, let B divide AC in ratio $k : 1$ (from A to C). Using the x-coordinate: $5 = \dfrac{k(11)+1(1)}{k+1} \implies 5(k+1) = 11k+1 \implies 5k+5 = 11k+1 \implies 6k = 4$

$$k = \frac{2}{3}$$

$$\boxed{B \text{ divides } AC \text{ in the ratio } 2:3}$$

Question 5. Show that the points A, B, C with position vectors $\vec{a}-2\vec{b}+3\vec{c}$, $2\vec{a}+3\vec{b}-4\vec{c}$, $-7\vec{b}+10\vec{c}$ are collinear.

Solution: Let position vectors be $\vec{p}_A = \vec{a} - 2\vec{b} + 3\vec{c}$, $\vec{p}_B = 2\vec{a} + 3\vec{b} - 4\vec{c}$, $\vec{p}_C = -7\vec{b} + 10\vec{c}$.

$$\overrightarrow{AB} = \vec{p}_B - \vec{p}_A = \vec{a} + 5\vec{b} - 7\vec{c}$$

$$\overrightarrow{AC} = \vec{p}_C - \vec{p}_A = -\vec{a} - 5\vec{b} + 7\vec{c} = -(\vec{a} + 5\vec{b} - 7\vec{c})$$

So $\overrightarrow{AC} = -\overrightarrow{AB}$, meaning $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are parallel and share point A. Therefore $\boxed{A, B, C \text{ are collinear}}$.

Question 6. Let $\vec{a} = \hat{\imath} + 4\hat{\jmath} + 2\hat{k}$, $\vec{b} = 3\hat{\imath} - 2\hat{\jmath} + 7\hat{k}$, $\vec{c} = 2\hat{\imath} - \hat{\jmath} + 4\hat{k}$. Find a vector $\vec{d}$ which is perpendicular to both $\vec{a}$ and $\vec{c}$, and satisfies $\vec{d} \cdot \vec{b} = 15$.

Solution: Since $\vec{d} \perp \vec{a}$ and $\vec{d} \perp \vec{c}$, $\vec{d}$ is parallel to $\vec{a} \times \vec{c}$:

$$\vec{a} \times \vec{c} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 4 & 2 \\ 2 & -1 & 4 \end{vmatrix}$$

$$\begin{aligned} &= \hat{\imath}(4 \cdot 4 - 2 \cdot (-1)) - \hat{\jmath}(1 \cdot 4 - 2 \cdot 2) + \hat{k}(1 \cdot (-1) - 4 \cdot 2) \\ &= \hat{\imath}(16+2) - \hat{\jmath}(4-4) + \hat{k}(-1-8) = 18\hat{\imath} + 0\hat{\jmath} - 9\hat{k} = 9(2\hat{\imath} - \hat{k}) \end{aligned}$$

So let $\vec{d} = t(2\hat{\imath} - \hat{k})$ for some scalar t. Using $\vec{d} \cdot \vec{b} = 15$:

$$t(2\hat{\imath} - \hat{k}) \cdot (3\hat{\imath} - 2\hat{\jmath} + 7\hat{k}) = t\big[2(3) + 0(-2) + (-1)(7)\big] = t(6-7) = -t$$

$$-t = 15 \implies t = -15$$

$$\vec{d} = -15(2\hat{\imath} - \hat{k}) = \boxed{-30\hat{\imath} + 15\hat{k}}$$

Check: $\vec{d} \cdot \vec{a} = -60 + 0 + 30 = 0\checkmark$, $\quad \vec{d} \cdot \vec{c} = -60 + 0 + 60 = 0\checkmark$, $\quad \vec{d} \cdot \vec{b} = -90 + 0 + 105 = 15\checkmark$

Question 7. Let $\vec{a}, \vec{b}, \vec{c}$ be the position vectors of points A, B, C satisfying $3\vec{a} - 2\vec{b} - \vec{c} = \vec{0}$. Show that A, B, C are collinear and find the ratio in which B divides AC.

Solution: From $3\vec{a} - 2\vec{b} - \vec{c} = \vec{0}$, we get $\vec{c} = 3\vec{a} - 2\vec{b}$. Then

$$\overrightarrow{BC} = \vec{c} - \vec{b} = 3\vec{a} - 2\vec{b} - \vec{b} = 3\vec{a} - 3\vec{b} = -3(\vec{b} - \vec{a}) = -3\,\overrightarrow{AB}$$

Since $\overrightarrow{BC}$ is a scalar multiple of $\overrightarrow{AB}$, the vectors are parallel and share the point B, so $\boxed{A, B, C \text{ are collinear}}$.
Since $\overrightarrow{BC} = -3\,\overrightarrow{AB}$, we have $|\overrightarrow{BC}| = 3|\overrightarrow{AB}|$ with B lying outside segment AC (the negative sign indicates external division).

$$\boxed{B \text{ divides } AC \text{ externally in the ratio } 1:3}$$

Question 8. If $\vec{a}, \vec{b}, \vec{c}$ are mutually perpendicular unit vectors, find $|\vec{a} + \vec{b} + \vec{c}|$.

Solution: Since the vectors are mutually perpendicular, $\vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{c} = \vec{c}\cdot\vec{a} = 0$. Since they are unit vectors, $|\vec{a}| = |\vec{b}| = |\vec{c}| = 1$.

$$|\vec{a} + \vec{b} + \vec{c}|^2 = |\vec{a}|^2 + |\vec{b}|^2 + |\vec{c}|^2 + 2(\vec{a}\cdot\vec{b} + \vec{b}\cdot\vec{c} + \vec{c}\cdot\vec{a})$$

$$= 1 + 1 + 1 + 2(0) = 3$$

$$\boxed{|\vec{a} + \vec{b} + \vec{c}| = \sqrt{3}}$$

Quick Revision – Formula Sheet

Everything from Chapter 10 in one place

1. **Position vector** of $P(x, y, z)$: $\overrightarrow{OP} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$, $|\overrightarrow{OP}| = \sqrt{x^2 + y^2 + z^2}$.
2. **Unit vector:** $\hat{a} = \dfrac{\vec{a}}{|\vec{a}|}$.
3. **Direction cosines** (l, m, n): $l^2 + m^2 + n^2 = 1$.
4. **Vector joining two points** P, Q: $\overrightarrow{PQ} = \vec{q} - \vec{p}$ (position vector of Q minus that of P).
5. **Section formula:** internal division in ratio $m : n$: $\vec{r} = \dfrac{m\vec{q} + n\vec{p}}{m + n}$; external division: $\vec{r} = \dfrac{m\vec{q} - n\vec{p}}{m - n}$.
6. **Collinearity test:** $\overrightarrow{AB} = k\,\overrightarrow{AC}$ for some scalar $k \iff A, B, C$ collinear.
7. **Scalar (dot) product:** $\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos\theta = a_1b_1 + a_2b_2 + a_3b_3$; $\vec{a} \perp \vec{b} \iff \vec{a} \cdot \vec{b} = 0$.
8. **Projection** of $\vec{a}$ on $\vec{b}$: $\dfrac{\vec{a} \cdot \vec{b}}{|\vec{b}|}$.
9. **Vector (cross) product:** $\vec{a} \times \vec{b} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = |\vec{a}||\vec{b}| \sin\theta\, \hat{n}$; $\vec{a} \parallel \vec{b} \iff \vec{a} \times \vec{b} = \vec{0}$.
10. **Area of triangle** with sides $\vec{a}, \vec{b}$: $\frac{1}{2}|\vec{a} \times \vec{b}|$. **Area of parallelogram:** $|\vec{a} \times \vec{b}|$.
11. **Key identities:** $\vec{a} \times \vec{a} = \vec{0}$; $\vec{b} \times \vec{a} = -\vec{a} \times \vec{b}$; $\vec{a} \cdot \vec{a} = |\vec{a}|^2$; $|\vec{a} + \vec{b}|^2 = |\vec{a}|^2 + |\vec{b}|^2 + 2\,\vec{a} \cdot \vec{b}$.

End of Chapter 10: Vector Algebra

All exercises (10.1 – 10.4) and the Miscellaneous Exercise are covered above with complete, verified, step-by-step solutions.